An iterative reconstruction of source functions in a potential problem using the MFS

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In this work, we address the reconstruction of characteristic source functions in a potential problem, from the knowledge of full and partial boundary data. The inverse problem is formulated as an inverse obstacle problem and two iterative methods are applied. A decomposition method based on the Kirsch-Kress method (requires Cauchy data reconstruction) and a Newton-type of method based on the domain derivative (requires the resolution of direct transmission problems). For the reconstruction of Cauchy data we use the method of fundamental solutions (MFS) and we show that, for partial data, we can consider only one exterior artificial boundary. We test the domain derivative method using the MFS (for transmission problems) and present theoretical results that justifies this numerical approximation. The feasibility of these methods will be illustrated by numerical simulations for both full and partial data.

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1. Introduction

The identification and reconstruction of source functions from boundary data is an inverse problem with many applications in nondestructive testing (eg. [1]). It is well known that, for a potential problem, a general source function can not be identified from boundary data (see [2], [3] for an overview on the subject). Several works have addressed the identification of particular source functions. For instance, point sources (eg. [3], [5]), surface sources (eg. [4]), source functions with known Laplacian (cf. [6]), characteristic source functions (cf. [7]). More generally, the same type of limitations on the identification occur on inverse acoustic source problems. In this case, however, the source can be identified from an infinite number of boundary measurements by varying the wavenumber (cf. [8]).

In this work, we address the identification and reconstruction of characteristic source functions $f = \chi_\omega$ defined on a 2D star shaped domain $\omega \subset\subset \Omega$, from the knowledge of full and partial boundary data on $\Sigma \subseteq \partial\Omega$. This ill-posed and non linear problem has been addressed in [9] where an iterative method was implemented via domain derivative, using the boundary element method (BEM) as numerical solver. More recently, a method based on the reciprocity gap functional

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was proposed and tested in [10]. In both cases, the study was only for full boundary data.

We propose two approaches for the shape reconstruction: First, a decomposition method based on the Kirsch-Kress method (cf. [11]). Second, the domain derivative method proposed in [9] but using the MFS as numerical solver. The first approach requires, at a first step, the reconstruction of Cauchy data using the MFS (cf. [12] for an overview and [13] for theoretical results on the MFS for inverse problems). The second approach requires the MFS for (direct) transmission problems.

One of the main advantages of the MFS for direct problems (eg. [14]) is that provides fast and accurate results (at least for smooth boundaries and smooth data) and has small implementation costs. It is also a meshfree method and is particular suited for iterative methods that requires solving several direct problems per iteration. Recently, the MFS approximation for a transmission potential problem have been successfully implemented (cf. [15]) and we present some theoretical results that justifies this numerical method.

This work is the result of the proceedings paper [16] and is organized as follows: In section 3 we present some results that justifies the MFS approximation for direct transmission problems. Section 4 describes the decomposition method and we prove that for partial boundary data, the MFS for Cauchy data fitting requires only one exterior artificial boundary. Section 5 concerns the iterative method using domain derivative. Last section concerns numerical simulations and discussion. We start with the direct and inverse problems formulation.

2. Direct and inverse problems

Let Ω be a domain in $\mathbb{R}^2$, that is, an open, bounded and simply connected set with regular boundary $\Gamma := \partial\Omega$ (at least C^1). Denote by χ_ω the characteristic function

$$\chi_\omega(x) = \begin{cases} 1 & \text{if } x \in \omega \\ 0 & \text{otherwise} \end{cases}$$

on a domain $\omega \subset\subset \Omega$ with (regular) boundary $\gamma := \partial\omega$.

The **direct problem** is, given a characteristic source function χ_ω and an input function g_Γ on Γ, compute $g_\Gamma^{\mathbf{n}} := \partial_{\mathbf{n}} u$ on Γ, where u solves

$$\begin{cases} \Delta u = \chi_\omega & \text{in } \Omega \\ u = g_\Gamma & \text{on } \Gamma \end{cases}.$$

Since $\chi_\omega \in L^2(\Omega)$ and considering $g_\Gamma \in H^{1/2}(\Gamma)$ then this problem is well posed in $H^1(\Omega)$. Define

$$H_\Gamma := H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma).$$

In this setting, the pair of compatible Cauchy data $(g_\Gamma, g_\Gamma^{\mathbf{n}})$ belongs to H_Γ.

Note that the above direct problem can also be formulated as the transmission problem

$$\begin{cases} \Delta u^+ = 0 & \text{in } \Omega_c := \Omega \setminus \overline{\omega} \\ u^+ = g_\Gamma & \text{on } \Gamma \\ [u] = 0 & \text{on } \gamma \\ [\partial_{\mathbf{n}} u] = 0 & \text{on } \gamma \\ \Delta u^- = 1 & \text{in } \omega \end{cases}$$

where $[u] = u^+ - u^-$ and $[\partial_{\mathbf{n}} u] = \partial_{\mathbf{n}} u^+ - \partial_{\mathbf{n}} u^-$ are the jump of the trace and normal trace across γ, respectively. We assume that the normal direction on γ points outwards with respect to Ω_c (hence, inwards with respect to ω).

Let u_p be a function that satisfies $\Delta u_p = 1$ in ω (take for instance $u_p(x) = x.x/4$). Then, considering

$$\begin{cases} \Delta u_H^+ = 0 & \text{in } \Omega_c := \Omega \setminus \overline{\omega} \\ u_H^+ = g_\Gamma & \text{on } \Gamma \\ [u_H] = u_p & \text{on } \gamma \\ [\partial_{\mathbf{n}} u_H] = \partial_{\mathbf{n}} u_p & \text{on } \gamma \\ \Delta u_H^- = 0 & \text{in } \omega \end{cases}$$

we have

$$(u^+, u^-) = (u_H^+, u_H^- - u_p).$$

We shall also consider the more general transmission problem

$$(T) \begin{cases} \Delta u^+ = 0 & \text{in } \Omega_c := \Omega \setminus \overline{\omega} \\ u^+ = g_\Gamma & \text{on } \Gamma \\ [u] = g_\gamma & \text{on } \gamma \\ [\partial_{\mathbf{n}} u] = g_\gamma^{\mathbf{n}} & \text{on } \gamma \\ \Delta u^- = 0 & \text{in } \omega \end{cases} .$$

This problem is well posed in $H^1(\Omega_c) \times H^1(\omega)$, taking the input functions $(g_\Gamma, g_\gamma, g_\gamma^{\mathbf{n}})$ in $H^{1/2}(\Gamma) \times H_\gamma$.

The **inverse problem** is to retrieve χ_ω (or equivalently the shape of ω) from a pair of compatible Cauchy boundary data $(g_\Gamma, g_\Gamma^{\mathbf{n}})$. We will also consider the inverse problem for partial boundary data. This problem can be formulated as follows: Consider the disjoint union

$$\Gamma = \Sigma \cup \Pi \cup \left(\Gamma \setminus \overline{\Sigma}\right)$$

where, Σ is a relatively open (non-empty) subset of Γ such that $\Gamma \setminus \overline{\Sigma} \neq \emptyset$. Define the spaces

$$H^{1/2}(\Sigma) := \left\{ u|_\Sigma : u \in H^{1/2}(\Gamma) \right\}$$

and

$$\widetilde{H}^{1/2}(\Sigma) := \left\{ u \in H^{1/2}(\Sigma) : \text{supp } u \subset \overline{\Sigma} \right\}.$$

Let

$$H^{-1/2}(\Sigma) := \left(\widetilde{H}^{1/2}(\Sigma)\right)'$$

and

$$\widetilde{H}^{-1/2}(\Sigma) := \left(H^{1/2}(\Sigma)\right)'$$

be the dual space of $\widetilde{H}^{1/2}(\Sigma)$ and $H^{1/2}(\Sigma)$, respectively (see [17]). The inverse problem for partial boundary data is to retrieve the characteristic function χ_ω from a pair of Cauchy data $(g_\Sigma, g_\Sigma^{\mathbf{n}}) \in H_\Sigma := H^{1/2}(\Sigma) \times H^{-1/2}(\Sigma)$.

2.1. *Identification*

It is well known that the Dirichlet-to-Neumann map for this problem can be obtained by a single measurement and that, in general, it is not possible to recover a characteristic function from this data (eg. [9]). However, for characteristic functions defined on *star shaped* domains we have the following identification result, due to Novikov.

Theorem 2.1: *[7] Let $\omega \subset\subset \Omega$ be a star shaped domain. Then, a single pair of full compatible Cauchy data $(g_\Gamma, g_\Gamma^{\mathbf{n}})$ determines uniquely χ_ω.*

Since $\text{supp}\chi_\omega \subset\subset \Omega$ then Holmgren's lemma and the above theorem provides the following identification from partial data.

Proposition 2.2: *A single pair of partial compatible Cauchy data $(g_\Sigma, g_\Sigma^{\mathbf{n}})$ determines uniquely a characteristic function defined on a star shaped domain.*

Proof: We follow [8]. Suppose that exists $u \in H^1(\Omega)$ satisfying

$$\begin{cases} \Delta u = \chi_\sigma & \text{in } \Omega \\ u = 0 & \text{on } \Sigma \\ \partial_{\mathbf{n}} u = 0 & \text{on } \Sigma \end{cases}$$

where $\sigma \subset\subset \Omega$ is any domain. Then, $\Delta u = 0$ in the open set $\Omega \setminus \overline{\sigma}$. Since $\Sigma \subset \Gamma \subset \partial(\Omega \setminus \overline{\sigma})$ and $(u|_\Sigma, \partial_{\mathbf{n}} u|_\Sigma) = (0,0)$ then, by Holmgren's lemma, $u = 0$ in $\Omega \setminus \overline{\sigma}$. This implies $(u|_\Gamma, \partial_{\mathbf{n}} u|_\Gamma) = (0,0)$. In particular, if $\chi_{\omega_1},\ \chi_{\omega_2}$ are characteristic functions defined on star shaped domains $\omega_1, \omega_2 \subset\subset \Omega$ generating boundary data $(g_\Gamma^1, g_\Gamma^{\mathbf{n},1})$ and $(g_\Gamma^2, g_\Gamma^{\mathbf{n},2})$, respectively, with $g_\Sigma^1 = g_\Sigma^2$ and $g_\Sigma^{\mathbf{n},1} = g_\Sigma^{\mathbf{n},2}$ then we must have $(g_\Gamma^1, g_\Gamma^{\mathbf{n},1}) = (g_\Gamma^2, g_\Gamma^{\mathbf{n},2})$ and the identification follows from the above Theorem. □

3. The MFS for transmission problems

Consider some 2D regular domains $\widehat{\Omega}$, $\widehat{\omega}_i$, $\widehat{\omega}_e$ such that $\Omega \subset\subset \widehat{\Omega}$, $\widehat{\omega}_i \subset\subset \omega$ and $\omega \subset\subset \widehat{\omega}_e$. Define $\widehat{\Omega}_c = \widehat{\Omega} \setminus \overline{\widehat{\omega}_i}$ and note that $\Omega_c \subset\subset \widehat{\Omega}_c$. Following the above notation, we define the boundaries $\widehat{\Gamma} := \partial\widehat{\Omega}$, $\widehat{\gamma}_i := \partial\widehat{\omega}_i$ and $\widehat{\gamma}_e := \partial\widehat{\omega}_e$ (see Fig. 1).

On these artificial boundaries, we define the following layer representations

$$u_1(x) = SL_{\widehat{\Gamma}}(\phi)(x) + SL_{\widehat{\gamma}_i}(\varphi)(x),\ x \in \mathbb{R}^2 \setminus (\widehat{\Gamma} \cup \widehat{\gamma}_i) \tag{1}$$

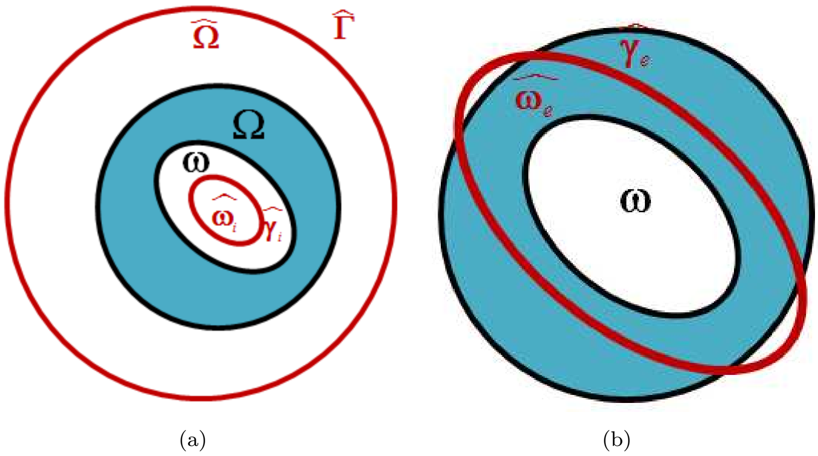


Figure 1. The artificial domains $\widehat{\Omega}$, $\widehat{\omega}_i$ and $\widehat{\omega}_e$.

$$u_2(x) = SL_{\widehat{\gamma}_e}(\psi)(x), \ x \in \mathbb{R}^2 \setminus \widehat{\gamma}_e. \tag{2}$$

in terms of single layer potentials

$$SL_{\widehat{\Gamma}}(\phi)(x) := \int_{\widehat{\Gamma}} \Phi_x(y)\phi(y)d\sigma_y.$$

Here,

$$\Phi_y(x) := -\frac{1}{2\pi}\log|x-y|$$

is a fundamental solution centered at y, that is, $\Delta\Phi_y = -\delta_y$ and δ is the Dirac delta.

Note that

$$\Delta u_1 = 0 \text{ in } \Omega_c \text{ and } \Delta u_2 = 0 \text{ in } \omega. \tag{3}$$

Thus, to solve (T) using this representation we have to solve the following system of integral equations

$$\left\{\begin{array}{l} SL_{\Gamma,\widehat{\Gamma}}(\phi) + SL_{\Gamma,\widehat{\gamma}_i}(\varphi) = g_\Gamma \\ SL_{\gamma,\widehat{\Gamma}}(\phi) + SL_{\gamma,\widehat{\gamma}_i}(\varphi) - SL_{\gamma,\widehat{\gamma}_e}(\psi) = g_\gamma \\ SL^{\mathbf{n}}_{\gamma,\widehat{\Gamma}}(\phi) + SL^{\mathbf{n}}_{\gamma,\widehat{\gamma}_i}(\varphi) - SL^{\mathbf{n}}_{\gamma,\widehat{\gamma}_e}(\psi) = g^{\mathbf{n}}_\gamma \end{array}\right. ,$$

where $SL_{\Gamma,\widehat{\Gamma}}$ and $SL^{\mathbf{n}}_{\Gamma,\widehat{\Gamma}}$ are the trace and normal trace of $SL_{\widehat{\Gamma}}$ on Γ, respectively.

This system is solvable for input functions $(g_\Gamma, g_\gamma, g^{\mathbf{n}}_\gamma)$ in the range of the linear and bounded operator $\mathcal{M} : H^{-1/2}(\widehat{\Gamma}) \times H^{-1/2}(\widehat{\gamma}_i) \times H^{-1/2}(\widehat{\gamma}_e) \to H^{1/2}(\Gamma) \times H_\gamma$ given by

$$\mathcal{M}(\phi,\varphi,\psi) := \begin{bmatrix} SL_{\Gamma,\widehat{\Gamma}} & SL_{\Gamma,\widehat{\gamma}_i} & 0 \\ SL_{\gamma,\widehat{\Gamma}} & SL_{\gamma,\widehat{\gamma}_i} & -SL_{\gamma,\widehat{\gamma}_e} \\ SL^{\mathbf{n}}_{\gamma,\widehat{\Gamma}} & SL^{\mathbf{n}}_{\gamma,\widehat{\gamma}_i} & -SL^{\mathbf{n}}_{\gamma,\widehat{\gamma}_e} \end{bmatrix} \begin{bmatrix} \phi \\ \varphi \\ \psi \end{bmatrix}.$$

However, the above operator is not surjective. Instead, we prove that the restriction of $\mathcal{M}$ to an appropriate space is injective and has dense range. In our framework, the previous restriction on the functional space is only needed for the 2D case, because of the asymptotic behavior of the fundamental solution. Consider the space

$$\mathcal{H}^{\pm 1/2}(\Gamma) := \left\{ u \in H^{\pm 1/2}(\Gamma) : \int_\Gamma u = 0 \right\} \simeq H^{\pm 1/2}(\Gamma)/\mathbb{R}$$

where, for $\mathcal{H}^{-1/2}(\Gamma)$ the above integral must be interpreted in the duality sense.

Proposition 3.1: *The restriction of $\mathcal{M}$ to $\mathcal{H}^{-1/2}(\widehat{\Gamma}) \times \mathcal{H}^{-1/2}(\widehat{\gamma}_i) \times \mathcal{H}^{-1/2}(\widehat{\gamma}_e)$ is injective.*

Proof: Consider the above single layer representations (1) and (2) u_1, u_2, for some densities $(\phi, \varphi, \psi) \in \ker \mathcal{M}$. This last condition and equations (3) imply that the pair (u_1, u_2) solves the transmission problem (T) for null input data. Thus $u_1 = 0$ in Ω_c and $u_2 = 0$ in ω. By analytic continuation, the traces (coming from $\widehat{\Omega}_c$) $u_1|_{\partial\widehat{\Omega}_c}$ and $\partial_{\mathbf{n}} u_1|_{\partial\widehat{\Omega}_c}$ are null. In the same way, the traces (coming from inside $\widehat{\gamma}_e$) $u_2|_{\partial\widehat{\omega}_e}$ and $\partial_{\mathbf{n}} u_2|_{\partial\widehat{\omega}_e}$ are null. Since the single layer representation implies the continuity of the trace across the boundary we must have $u_1|_{\partial\widehat{\Omega}_c} = 0$ and $u_2|_{\partial\widehat{\omega}_e} = 0$ coming from outside of $\widehat{\Omega}_c$ and $\widehat{\omega}_e$, respectively. Now, since the exterior problems

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2 \setminus \overline{\widehat{\Omega}} \\ u = 0 & \text{on } \widehat{\Gamma} \\ u = O(1) & |x| \to \infty \end{cases} , \quad \begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^2 \setminus \overline{\widehat{\omega}}_e \\ v = 0 & \text{on } \widehat{\gamma}_e \\ v = O(1) & |x| \to \infty \end{cases} ,$$

$(u, v) \in H^1_{loc}(\mathbb{R}^2 \setminus \overline{\widehat{\Omega}}) \times H^1_{loc}(\mathbb{R}^2 \setminus \overline{\widehat{\omega}}_e)$, are well posed and $(\phi, \varphi, \psi) \in \mathcal{H}^{-1/2}(\widehat{\Gamma}) \times \mathcal{H}^{-1/2}(\widehat{\gamma}_i) \times \mathcal{H}^{-1/2}(\widehat{\gamma}_e)$ then, (u_1, u_2) satisfy the above problems. It follows that the normal traces $\partial_{\mathbf{n}} u_1|_{\widehat{\Gamma}}$ (coming from outside of $\widehat{\Omega}_c$) and $\partial_{\mathbf{n}} u_2|_{\widehat{\gamma}_e}$ (coming from outside of $\widehat{\omega}_e$) are null and therefore the normal jumps of u_1 and u_2 across $\widehat{\Gamma}$ and $\widehat{\gamma}_e$, respectively, are null. It is easy to establish that in the above conditions, the normal jump of u_1 across $\widehat{\gamma}_i$ is also null, from where it follows that

$$\phi = \varphi = \psi = 0.$$

□

The following lemma will be usefull to prove the denseness of the range.

Lemma 3.2: *The adjoint of $\mathcal{M}$ is the operator $\mathcal{M}^* : H^{-1/2}(\Gamma) \times (H_\gamma)' \to H^{1/2}(\widehat{\Gamma}) \times H^{1/2}(\widehat{\gamma}_i) \times H^{1/2}(\widehat{\gamma}_e)$ given by*

$$\mathcal{M}^* = \begin{bmatrix} SL_{\widehat{\Gamma},\Gamma} & SL_{\widehat{\Gamma},\gamma} & DL_{\widehat{\Gamma},\gamma} \\ SL_{\widehat{\gamma}_i,\Gamma} & SL_{\widehat{\gamma}_i,\gamma} & DL_{\widehat{\gamma}_i,\gamma} \\ 0 & -SL_{\widehat{\gamma}_e,\gamma} & -DL_{\widehat{\gamma}_e,\gamma} \end{bmatrix}$$

where $DL_{\widehat{\Gamma},\gamma}$ is the trace on $\widehat{\Gamma}$ of the double layer potential DL_γ.

Proof: We recall that the double layer potential is defined by

$$DL_\gamma(\phi)(x) = \int_\gamma \partial_{\mathbf{n}_y} \Phi_x(y) \phi(y) d\sigma_y.$$

The result follows from the well known properties of the single and double layer potentials (eg [17]). □

Proposition 3.3: *The restriction of $\mathcal{M}$ to $\mathcal{H}^{-1/2}(\widehat{\Gamma})\times\mathcal{H}^{-1/2}(\widehat{\gamma}_i)\times\mathcal{H}^{-1/2}(\widehat{\gamma}_e)$ has dense range.*

Proof: It is sufficient to prove that the restriction of $\mathcal{M}^*$ to $\mathcal{H}^{-1/2}(\Gamma)\times\mathcal{H}^{-1/2}(\gamma)\times H^{1/2}(\gamma)$ is injective. Let $(\phi,\varphi,\psi)\in\ker\mathcal{M}^*$ and consider the function

$$u_1 = SL_\Gamma(\phi) + SL_\gamma(\varphi) + DL_\gamma(\psi).$$

By hypothesis, $u_1 = 0$ on $\widehat{\Gamma}\cup\widehat{\gamma}_i$ and since $\Delta u_1 = 0$ in $\mathbb{R}^2\setminus(\Gamma\cup\gamma)$ then using the arguments of the previous proposition we must have

$$u_1|_{\partial\Omega_c} = \partial_{\mathbf{n}} u_1|_{\partial\Omega_c} = 0$$

(coming from the exterior of Ω_c). Thus,

$$\phi = [\partial_{\mathbf{n}} u_1]_\Gamma = \partial_{\mathbf{n}} u_1|_\Gamma,\ \varphi = [\partial_{\mathbf{n}} u_1]_\gamma = \partial_{\mathbf{n}} u_1|_\gamma,\ -\psi = [u_1]_\gamma = u_1|_\gamma$$

where the traces are taken coming from the interior of Ω_c.

Now, define the function (we are still assuming that the normal direction points inwards to ω)

$$u_2 = SL_\gamma(-\varphi) + DL_\gamma(-\psi).$$

By hypothesis, $u_2 = 0$ in $\widehat{\gamma}_e$ and by analytic continuation arguments, we must have $u_2 = 0$ in $\mathbb{R}^2\setminus\overline{\omega}$. From this,

$$\varphi = [\partial_{\mathbf{n}} u_2]_\gamma = \partial_{\mathbf{n}} u_2|_\gamma,\ -\psi = [u_2]_\gamma = u_2|_\gamma,$$

taking the traces coming from inside of ω. Therefore,

$$\begin{cases} u_1|_\Gamma = 0 \\ u_1|_\gamma - u_2|_\gamma = -\psi - (-\psi) = 0 \\ \partial_{\mathbf{n}} u_1|_\gamma - \partial_{\mathbf{n}} u_2|_\gamma = \varphi - \varphi = 0 \end{cases}$$

and the pair (u_1,u_2) solves problem (T) with null input data hence, $u_1 = 0$ in Ω_c and $u_2 = 0$ in ω. We conclude that $u_1 = 0$ in $\mathbb{R}^2\setminus\partial\Omega_c$ and $u_2 = 0$ in $\mathbb{R}^2\setminus\partial\omega$ and therefore the jumps ϕ,φ,ψ must be null. □

In particular,

Corollary 3.4: *The space*

$$span\left(\left\{(\Phi_y|_\Gamma,\Phi_y|_\gamma,\partial_{\mathbf{n}}\Phi_y|_\gamma) : y\in\widehat{\Gamma}\cup\widehat{\gamma}_i\right\}\cup\{(0,\Phi_z|_\gamma,\partial_{\mathbf{n}}\Phi_z|_\gamma) : z\in\widehat{\gamma}_e\}\right)+\mathbb{R}$$

is dense in $H^{1/2}(\Gamma)\times H_\gamma$.

In order to construct the MFS approximation for the transmission problem, we consider (for simplicity, we shall drop the constants)

$$\widetilde{u}_1 = \underbrace{\sum_{j=1}^{m_1} \alpha_j \Phi(x - y_j^{\widehat{\Gamma}})}_{\widetilde{SL}_{\widehat{\Gamma}}(\alpha_1,\ldots,\alpha_{m_1})} + \underbrace{\sum_{j=m_1+1}^{m_2} \alpha_j \Phi(x - y_j^{\widehat{\gamma}_i})}_{\widetilde{SL}_{\widehat{\gamma}_i}(\alpha_{m_1+1},\ldots,\alpha_{m_2})}, \qquad \widetilde{u}_2 = \underbrace{\sum_{j=m_2+1}^{m} \alpha_j \Phi(x - y_j^{\widehat{\gamma}_e})}_{\widetilde{SL}_{\widehat{\gamma}_e}(\alpha_{m_2+1},\ldots,\alpha_m)} \quad (4)$$

where $y_j^{\widehat{\Gamma}}$ are source points on the artificial boundary curve $\widehat{\Gamma}$. These functions $\widetilde{u}_1$, $\widetilde{u}_2$ can be seen as a discretization of the single layer integrals (1) and (2) defining the representations u_1 and u_2. The corresponding discretization of $\mathcal{M}$ is $\widetilde{\mathcal{M}} : \mathbb{R}^m \to L^2(\Gamma) \times L^2(\gamma)^2$,

$$\widetilde{\mathcal{M}}(\alpha_1, \ldots, \alpha_m) := \begin{bmatrix} \widetilde{SL}_{\Gamma,\widehat{\Gamma}} & \widetilde{SL}_{\Gamma,\widehat{\gamma}_i} & 0 \\ \widetilde{SL}_{\gamma,\widehat{\Gamma}} & \widetilde{SL}_{\gamma,\widehat{\gamma}_i} & -\widetilde{SL}_{\gamma,\widehat{\gamma}_e} \\ \widetilde{SL}^{\mathbf{n}}_{\gamma,\widehat{\Gamma}} & \widetilde{SL}^{\mathbf{n}}_{\gamma,\widehat{\gamma}_i} & -\widetilde{SL}^{\mathbf{n}}_{\gamma,\widehat{\gamma}_e} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix}.$$

Lemma 3.5: *$\widetilde{\mathcal{M}}$ is injective and, in particular,*

$$\mathcal{S} = \Big\{ \big(\Phi_{y_j}|_\Gamma, \Phi_{y_j}|_\gamma, \partial_{\mathbf{n}}\Phi_{y_j}|_\gamma\big) : y_j \in \widehat{\Gamma} \cup \widehat{\gamma}_i, \ \ j = 1, \ldots, n \Big\}$$
$$\cup \{(0, \Phi_{z_k}|_\gamma, \partial_{\mathbf{n}}\Phi_{z_k}|_\gamma) : z_k \in \widehat{\gamma}_e, \ \ k = 1, \ldots, m\}$$

is linearly independent.

Proof: Take $(\alpha_1, \ldots, \alpha_m) \in \ker \widetilde{\mathcal{M}}$ and consider $(\widetilde{u}_1, \widetilde{u}_2)$ defined as in (4). Then, the pair $(\widetilde{u}_1, \widetilde{u}_2)$ solves (T) with null input data hence $\widetilde{u}_1 = 0$ in Ω_c and $\widetilde{u}_2 = 0$ in ω. By analytic continuation, $\widetilde{u}_1 = 0$ in $\mathbb{R}^2 \setminus \{y_j^{\widehat{\Gamma}}, y_j^{\widehat{\gamma}_i}\}$ and $\widetilde{u}_2 = 0$ in $\mathbb{R}^2 \setminus \{y_j^{\widehat{\gamma}_e}\}$. The result now follows from the fact that

$$\{\Phi(\bullet - y_j) : j = 1, \ldots, m\}$$

is linearly independent. □

The coefficients can be computed by imposing

$$\widetilde{\mathcal{M}}(\alpha_1, \ldots, \alpha_m)(x_j^\Gamma, x_j^\gamma, x_j^\gamma) = \begin{bmatrix} g_\Gamma(x_j^\Gamma) \\ g_\gamma(x_j^\gamma) \\ g_\gamma^{\mathbf{n}}(x_j^\gamma) \end{bmatrix}$$

on some collocation points $x_j^\Gamma \in \Gamma$, $x_j^\gamma \in \gamma$ or in a least squares sense.

4. Shape reconstruction using a decomposition method

In this section we introduce a method for the reconstruction of $\gamma = \partial\omega$ from the available Cauchy data on $\Sigma \subset \Gamma$. In a first step, we use an appropriate harmonic function, $\widetilde{u}_1$, to fit this data. Clearly, if $\widetilde{u}_1$ is harmonic in some domain $\Omega \setminus \overline{\sigma} \supset \Omega_c$ and $(\widetilde{u}_1|_\Sigma, \partial_{\mathbf{n}}\widetilde{u}_1|_\Sigma) = (g_\Sigma, g_\Sigma^{\mathbf{n}})$ then by analytic continuation and Holmgren's lemma, u_1 is the solution of the direct problem in Ω_c. However, such function might not exist on such domain, even for compatible Cauchy data. In order to properly deal with this ill-posed nature of the problem the fitting of the data must be considered using some sort of regularization technique (here we shall consider Tikhonov).

On a second (non linear) step we search for $\widetilde{\gamma}$ in a proper class of shapes, in order to minimize the norm of the residual

$$\left|\left|\partial_{\mathbf{n}}\widetilde{u}_2|_{\widetilde{\gamma}} - \partial_{\mathbf{n}}\widetilde{u}_1|_{\widetilde{\gamma}} + \partial_{\mathbf{n}}u_p|_{\widetilde{\gamma}}\right|\right|_2,$$

where

$$\begin{cases} \Delta\widetilde{u}_2 = 0 & \text{in } \widetilde{\omega} \\ \widetilde{u}_2 = \widetilde{u}_1 - u_p & \text{on } \widetilde{\gamma} = \partial\widetilde{\omega} \end{cases}$$

and $u_p(x) = x \cdot x/4$.

For the first step, we consider a representation of $\widetilde{u}_1$ in terms of a linear combination of fundamental solutions centered at some source points (MFS for Cauchy data) which we describe in the following section.

4.1. *The MFS applied to Cauchy data fitting*

4.1.1. Full data

Let u_1 be defined by equation (1). In order to fit the boundary data $(g_\Gamma, g_\Gamma^{\mathbf{n}}) \in H_\Gamma \left(= H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)\right)$ we consider the boundary integral equations $\mathcal{K}_\Gamma(\phi,\psi) = (g_\Gamma, g_\Gamma^{\mathbf{n}})$, where $\mathcal{K}_\Gamma : H^{1/2}(\widehat{\Gamma}) \times H^{1/2}(\widehat{\gamma}_i) \to H_\Gamma$ is the linear and bounded operator

$$\mathcal{K}_\Gamma(\phi,\psi) := \begin{bmatrix} SL_{\Gamma,\widetilde{\Gamma}} & SL_{\Gamma,\widehat{\gamma}_i} \\ SL^{\mathbf{n}}_{\Gamma,\widetilde{\Gamma}} & SL^{\mathbf{n}}_{\Gamma,\widehat{\gamma}_i} \end{bmatrix} \begin{bmatrix} \phi \\ \psi \end{bmatrix}.$$

We have the following result (cf. [18]).

Lemma 4.1: *The restriction of $\mathcal{K}_\Gamma$ to $\mathcal{H}^{1/2}(\widehat{\Gamma}) \times \mathcal{H}^{1/2}(\widehat{\gamma}_i)$ has dense range in H_Γ hence*

$$\mathit{span}\left\{(\Phi_y|_\Gamma, \partial_{\mathbf{n}}\Phi_y|_\Gamma) : y \in \widehat{\Gamma} \cup \widehat{\gamma}_i\right\} + \mathbb{R}$$

is dense in H_Γ.

Remark 1: In the previous density result, an interior artificial curve $\widehat{\gamma}_i \subset \omega$ must be considered. In particular, we must have some knowledge on the location and dimension of ω. This is not a big restriction since, as discussed in [10], the barycenter and measure of ω can be retrieved from the Cauchy data.

The above result justifies the MFS approximation for the Cauchy problem given by a linear combination of fundamental solutions centered at several source points $y_1, \ldots, y_m \in \widehat{\Gamma} \cup \widehat{\gamma}_i$, that is we consider the expansion

$$\widetilde{u}_1 := \sum_{i=1}^{m} \alpha_i \Phi_{y_i}, \quad y_i \in \widehat{\Gamma} \cup \widehat{\gamma}_i.$$

Lemma 4.2: *The set*

$$K = \left\{ (\Phi_{y_j}|_\Gamma, \partial_{\mathbf{n}} \Phi_{y_j}|_\Gamma) : y_j \in \widehat{\Gamma} \cup \widehat{\gamma}_i, \;\; j = 1, \ldots, m \right\}$$

is linearly independent.

Proof: Let $(\alpha_1, \ldots, \alpha_m)$ be such that

$$\sum_{i=1}^{m} \alpha_i \Phi_{y_i}(x) = 0 \quad \text{and} \quad \sum_{i=1}^{m} \alpha_i \partial_{\mathbf{n}} \Phi_{y_i}(x) = 0, \; \forall x \in \Gamma.$$

Defining $\widetilde{u}_1 = \sum_{i=1}^{m} \alpha_i \Phi_{y_i}$ we have $(\widetilde{u}_1|_\Gamma, \partial_{\mathbf{n}} \widetilde{u}_1|_\Gamma) = (0,0)$. On the other hand, $\Delta u = 0$ in Ω and by Holmgren's lemma and analytic continuation, $u = 0$ in $\mathbb{R}^2 \setminus \{y_1, \ldots, y_m\}$. Since $\{\Phi(\bullet - y_j) : j = 1, \ldots, m\}$ is independent, the result follows. □

We compute the coefficients in order to fit the boundary conditions on some collocation points $x_1, \ldots, x_n \in \Gamma$ $(n \geq m/2)$. Due to ill-conditioning problems, some regularization method must be considered and we apply the Tikhonov regularization method. Thus, we compute the coefficients α_i by solving the system

$$(\mu \mathbb{I} + \mathbb{A}^* \mathbb{A}) \mathbf{X} = \mathbb{A}^* \mathbb{B}$$

where $\mu > 0$ is the regularization parameter, $\mathbb{I}$ is the identity matrix,

$$\mathbb{A} = \begin{bmatrix} \Phi_{y_1}(x_1) & \ldots & \Phi_{y_m}(x_1) \\ \ldots & \ldots & \ldots \\ \Phi_{y_1}(x_n) & \ldots & \Phi_{y_m}(x_n) \\ \partial_{\mathbf{n}} \Phi_{y_1}(x_1) & \ldots & \partial_{\mathbf{n}} \Phi_{y_m}(x_1) \\ \ldots & \ldots & \ldots \\ \partial_{\mathbf{n}} \Phi_{y_1}(x_n) & \ldots & \partial_{\mathbf{n}} \Phi_{y_m}(x_n) \end{bmatrix} \text{ and } \mathbb{B} = \begin{bmatrix} g_\Gamma(x_1) \\ \ldots \\ g_\Gamma(x_n) \\ g_\Gamma^{\mathbf{n}}(x_1) \\ \ldots \\ g_\Gamma^{\mathbf{n}}(x_n) \end{bmatrix}.$$

4.1.2. Partial data

The above method can still be applied to fit partial Cauchy data. In this section we prove that for partial data we can apply the method considering only one exterior artificial boundary (albeit worst numerical results).

In the following, we shall drop the constants by taking an artificial domain (bounded, open and simply connected) $\widehat{\Omega}$ such that $\overline{\widehat{\Omega}} \subset \mathbb{R}^2 \setminus \overline{\Omega}$. Denote by $\widehat{\Gamma} = \partial \widehat{\Omega}$ his (regular) boundary and define the linear operator $\mathcal{K}_\Sigma : H^{-1/2}(\widehat{\Gamma}) \to H_\Sigma$ by

$$\mathcal{K}_\Sigma \phi = \left(SL_{\Sigma, \widehat{\Gamma}} \phi, SL^{\mathbf{n}}_{\Sigma, \widehat{\Gamma}} \phi \right)$$

where $SL_{\Sigma, \widehat{\Gamma}}$ is the restriction of $SL_{\Gamma, \widehat{\Gamma}}$ to Σ. We claim that this operator satisfies similar injective and denseness properties.

Lemma 4.3: $\mathcal{K}_\Sigma$ *is injective.*

Proof: Consider $\phi \in \ker \mathcal{K}_\Sigma$ and define

$$u_1 = SL_{\widehat{\Gamma}} \phi.$$

By hypothesis, the pair of Cauchy data $(u_1, \partial_{\mathbf{n}} u_1)|_\Sigma$ is null and since $\Delta u_1 = 0$ in $\mathbb{R}^2 \setminus \widehat{\Gamma}$ it follows that $u_1 = 0$ in $\mathbb{R}^2 \setminus \overline{\widehat{\Omega}}$. Since the single layer representation implies the continuity of the trace across the boundary $\widehat{\Gamma}$, the normal trace is also continuous across $\widehat{\Gamma}$ and we conclude that $\phi = 0$. □

Following the above section, before establishing the denseness result, we start by computing the adjoint of $\mathcal{K}_\Sigma$.

Lemma 4.4: *The adjoint of $\mathcal{K}_\Sigma$ is given by*

$$\mathcal{K}_\Sigma^*(\psi_1, \psi_2) = SL_{\widehat{\Gamma},\Sigma}\psi_1 + DL_{\widehat{\Gamma},\Sigma}\psi_2,\ (\psi_1, \psi_2) \in H_\Sigma^*$$

Proof: Let $\phi \in H^{-1/2}(\widehat{\Gamma})$ and $\psi = (\psi_1, \psi_2) \in H_\Sigma^* = \widetilde{H}^{-1/2}(\Sigma) \times \widetilde{H}^{1/2}(\Sigma)$.

$$\begin{aligned}
\langle \mathcal{K}_\Sigma \phi, \psi \rangle_{H_\Sigma \times H_\Sigma^*} &= \left\langle SL_{\Sigma,\widehat{\Gamma}}\phi, \psi_1 \right\rangle_{H^{1/2}(\Sigma)\times \widetilde{H}^{-1/2}(\Sigma)} + \left\langle SL^{\mathbf{n}}_{\Sigma,\widehat{\Gamma}}\phi, \psi_2 \right\rangle_{H^{-1/2}(\Sigma)\times \widetilde{H}^{1/2}(\Sigma)} \\
&= \int_\Sigma \int_{\widehat{\Gamma}} \Phi_y(x)\phi(x) d\sigma_x \psi_1(y) d\sigma_y + \int_\Sigma \int_{\widehat{\Gamma}} \partial_{\mathbf{n}_y}\Phi_y(x)\phi(x) d\sigma_x \psi_2(y) d\sigma_y \\
&= \int_{\widehat{\Gamma}} \int_\Sigma \Phi_y(x)\psi_1(y) d\sigma_y \phi(x) d\sigma_x + \int_{\widehat{\Gamma}} \int_\Sigma \partial_{\mathbf{n}_y}\Phi_y(x)\psi_2(y) d\sigma_y \phi(x) d\sigma_x \\
&= \int_{\widehat{\Gamma}} \int_\Sigma \left(\Phi_y(x)\psi_1(y) + \partial_{\mathbf{n}_y}\Phi_y(x)\psi_2(y)\right) d\sigma_y \phi(x) d\sigma_x \\
&= \left\langle \phi, SL_{\widehat{\Gamma},\Sigma}\psi_1 + DL_{\widehat{\Gamma},\Sigma}\psi_2 \right\rangle_{H^{-1/2}(\widehat{\Gamma})\times H^{1/2}(\widehat{\Gamma})}.
\end{aligned}$$

□

Proposition 4.5: *$\mathcal{K}_\Sigma$ has dense range and therefore,*

$$span\left\{(\Phi_y|_\Sigma, \partial_{\mathbf{n}}\Phi_y|_\Sigma) : y \in \widehat{\Gamma}\right\}$$

is dense in H_Σ.

Proof: Under the identification (see [17])

$$\widetilde{H}^{-1/2}(\Sigma) \simeq \left\{u \in H^{-1/2}(\Gamma) : \text{supp } u \subset \overline{\Sigma}\right\}$$

let $(\psi_1, \psi_2) \in \ker \mathcal{K}_\Sigma^*$ and consider

$$u_1 = SL_\Gamma \widetilde{\psi}_1 + DL_\Gamma \psi_2,$$

where $\widetilde{\psi}_1 \in H^{-1/2}(\Gamma)$ is the extension of ψ_1 by zero to the whole boundary, Γ. By hypothesis, $u_1 = 0$ on $\widehat{\Gamma}$ and $\Delta_1 u = 0$ in $\mathbb{R}^2 \setminus \Gamma$ implies $u_1 = 0$ in $\mathbb{R}^2 \setminus \overline{\Omega}$. Hence,

$$\widetilde{\psi}_1 = [\partial_{\mathbf{n}} u_1]|_\Gamma = \partial_{\mathbf{n}} u_1^-|_\Gamma \quad \text{and} \quad \psi_2 = -[u_1]|_\Gamma = -u_1^-|_\Gamma.$$

On the other hand $\psi_2 \in \widetilde{H}^{1/2}(\Sigma)$, therefore supp $\psi_2 \subset \overline{\Sigma}$ and we get

$$(\widetilde{\psi}_1, \psi_2) = (0, 0)$$

on the open (non empty) set $\Gamma \setminus \overline{\Sigma}$. The above jump identities gives

$$(u_1, \partial_{\mathbf{n}} u_1)|_{\Gamma\setminus\overline{\Sigma}} = (0, 0)$$

and by Holmgren's lemma, it follows $u_1 = 0$ in Ω hence, $\widetilde{\psi}_1 = \psi_2 = 0$. □

5. Shape reconstruction using the domain derivative

Instead of decomposing ill-posedness and non-linearity of the inverse problem as proposed in previous section, we now consider the inverse problem formulated as a non linear and ill-posed equation

$$F(\gamma) = g_\Gamma^{\mathbf{n}}$$

where $F : \mathcal{A} \to L^2(\Gamma)$ is defined by $F(\gamma_\alpha) = \partial_{\mathbf{n}} u_\alpha|_\Gamma$ and $u_\alpha \in H^1(\Omega)$ satisfies

$$\begin{cases} \Delta u_\alpha = \chi_{\omega_\alpha} & \text{in } \Omega \\ u_\alpha = g_\Gamma & \text{on } \Gamma \end{cases},$$

for $g_\Gamma \in H^{1/2}(\Gamma)$ and $\gamma_\alpha = \partial\omega_\alpha \in \mathcal{A}$ where

$$\mathcal{A} := \left\{\gamma = \partial\omega \in C^1 : \omega \subset\subset \Omega\right\}$$

denotes the set of admissible shapes.

By linearising the above equation, we obtain

$$F(\gamma_\alpha) + F'(\gamma_\alpha)h = g_\Gamma^{\mathbf{n}}.$$

Proposition 5.1: *[9] The Fréchet derivative of F at γ_α is given by*

$$F'(\gamma_\alpha)h = \partial_{\mathbf{n}} u'|_\Gamma$$

where $u' \in H_0^1(\Omega)$ satisfies

$$\begin{cases} \Delta u' = 0 & \text{in } \Omega \setminus \gamma_\alpha \\ [u'] = 0 & \text{on } \gamma_\alpha \\ [\partial_{\mathbf{n}} u'] = h \cdot \mathbf{n} & \text{on } \gamma_\alpha \end{cases} \tag{5}$$

From this, applying the Levenberg-Marquardt method we obtain the update step

$$\left(\mu\mathbb{I} + \mathbb{J}^\top\mathbb{J}\right) h = -\mathbb{J}^\top\left(F(\gamma_\alpha) - g_\Gamma^{\mathbf{n}}\right) \tag{6}$$

where $\mathbb{J} = [\partial_{\mathbf{n}} u']$ is the Jacobian matrix. Note that each iteration step requires several direct transmission problems. We apply the MFS, as described in section 3, to solve these problems.

6. Numerical Simulations

In the following numerical simulations, we considered $\Gamma = \partial B(0, 4)$ and two obstacles. First, a kite shaped domain ω_1, with boundary defined by the parametrization

$$t \in [0, 2\pi[\to \mathbb{R}^2, \quad t \mapsto \gamma_1(t) = \left(1.2\cos(t) + 0.9\cos^2(t), 2.0\sin(t)\right).$$

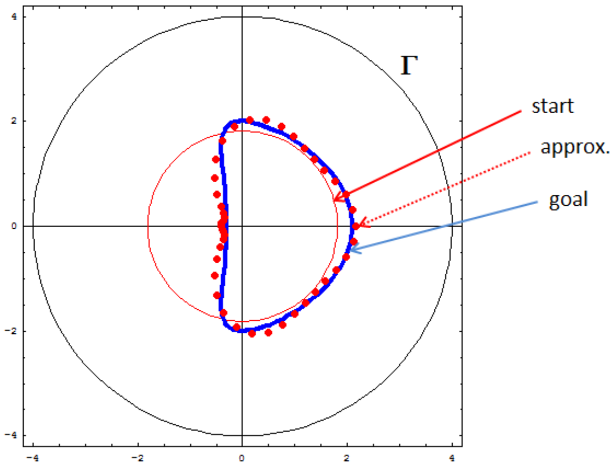


Figure 2. Reconstruction in $\mathcal{A}_4$ (full data without noise).

The second, ω_2, has a peanut shape and the boundary is given by the parametrization

$$t \in [0, 2\pi[\to \mathbb{R}^2, \quad t \mapsto \gamma_2(t) = \frac{2 + 0.9\cos(2t)}{2 + \sin^2(t)} (\cos(t), \sin(t)) + (1, 1).$$

For both cases, we took the input function $g_\Gamma \equiv 0$ and generated the measured data, $g_\Gamma^{\mathbf{n}}$, solving the corresponding direct problem with the MFS.

The first simulations, concerns the recovery of the kite using the decomposition method. Here, we considered the artificial boundaries $\widehat{\Gamma} = \partial B(0, 6)$ and $\widehat{\gamma}_1 = \partial B(0, 0.4)$ (these artificial boundaries are different from the ones considered in the direct problem). The number of observation points

$$g_\Gamma^{\mathbf{n}}(x_1), \ldots, g_\Gamma^{\mathbf{n}}(x_m)$$

were $m = 80$, uniformly distributed on the whole Γ (for full data). The Tikhonov regularization parameter was $\mu = 10^{-8}$. The starting shape was $\gamma^{(0)} = \partial B(0, 2.0)$ and we search for the updates in

$$\mathcal{A}_4 := \left\{ \sum_{j=0}^{4} (\alpha_j \cos(jt) + \alpha_{5+j} \sin(jt)) (\cos t, \sin t) \right\} \simeq \mathbb{R}^9.$$

The iterative procedure stops when the relative evolution of the objective function is $\leq 10^{-2}$. For data without noise, the procedure stopped after 6 iterations and the reconstruction result is presented in Fig. 2. In Fig. 3, we present the simulation results for the same setting but with data affected by (up to) 5% of random noise (this time, the procedure stopped after 10 iterations).

Next simulations (Fig. 4) concerns the reconstruction from partial boundary data with and without noise. In this case, we considered $m = 80$ observation points uniformly distributed over the arc (represented by the green dots)

$$\Sigma = \{4(\cos t, \sin t), t \in [0, \pi]\}.$$

Here, we observe a shadow effect, as the reconstruct is better in the arc where the observation points are located (recall that, by proposition 2.2, the obstacle can be

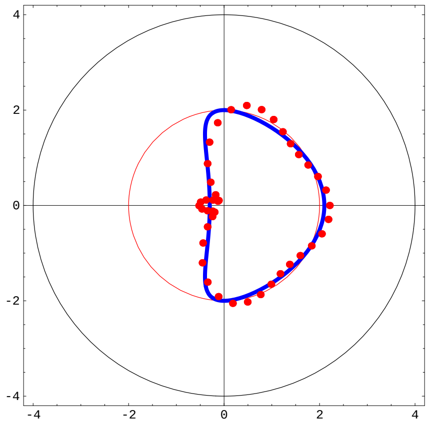
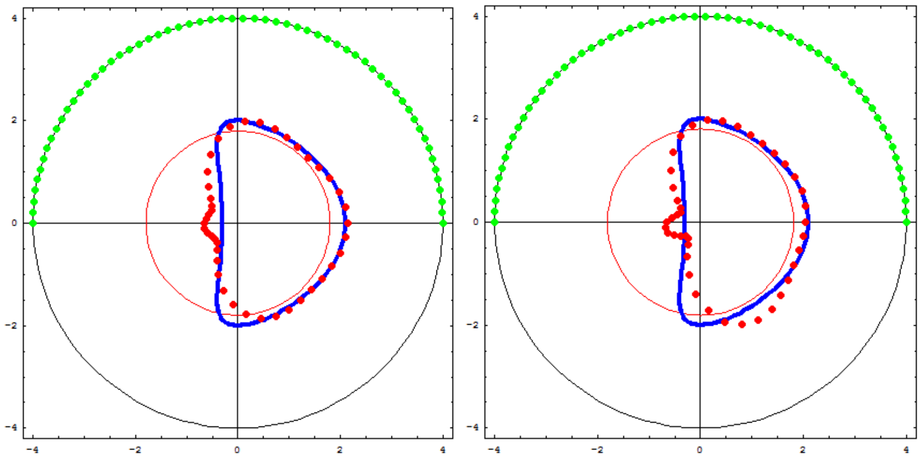


Figure 3. Reconstruction in $\mathcal{A}_4$ (full data with 5% of noise).



(a) (b)

Figure 4. Reconstruction in $\mathcal{A}_4$ from partial data without noise (left) and with 5 % of noise (right).

identified from partial boundary data). The number of iterations computed were 9 for both situations.

Regarding the reconstruction of the kite from full and partial data using the domain derivative we obtained the results plotted in Figs. 5 and 6. Here, it is more clear, the shadow effect in the reconstruction from partial data. This may be explained by different regularization effects of both methods. For instance, the parameter $\mu > 0$ in (6) is automatically controlled by the gain ratio (see [19] for details).

Last simulations regards the reconstruction of the peanut shaped obstacle using the domain derivative. The object is smaller than the kite and is not centered with the exterior boundary. The partial observation points are now in the arc

$$\Sigma = \left\{ 4(\cos t, \sin t),\ t \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \right\}.$$

Results of these simulations are presented in Figs. 7 and 8.

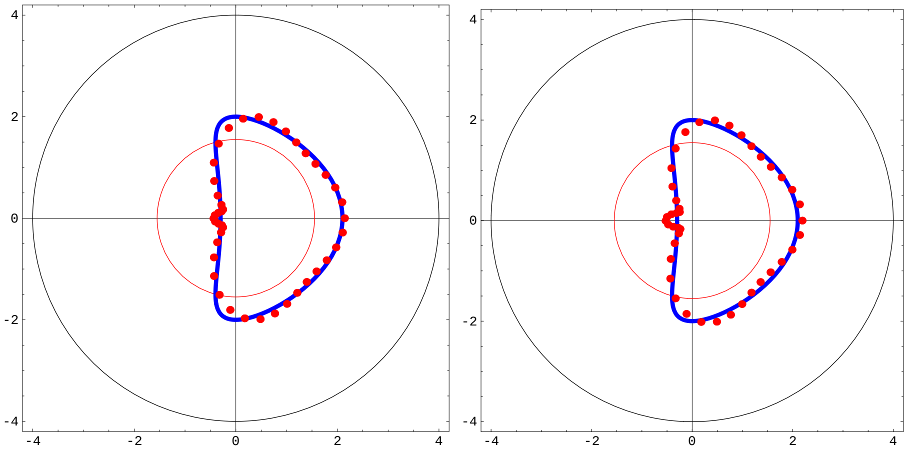


Figure 5. Reconstruction of a kite from full data without noise (left) and with 5 % of noise (right).

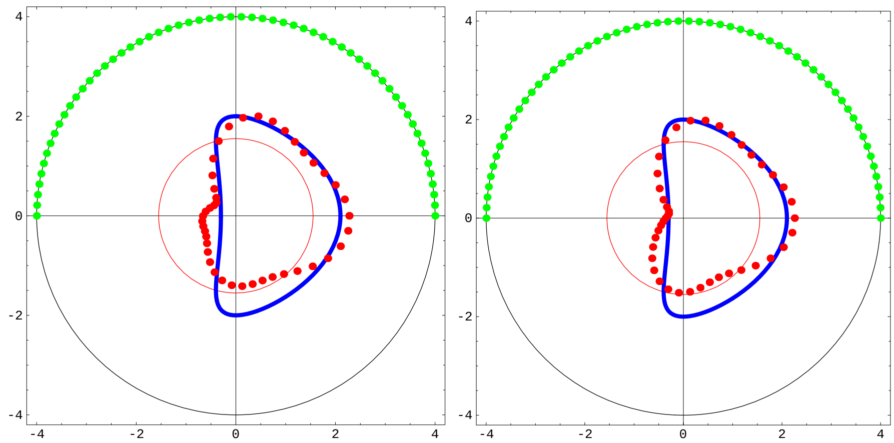
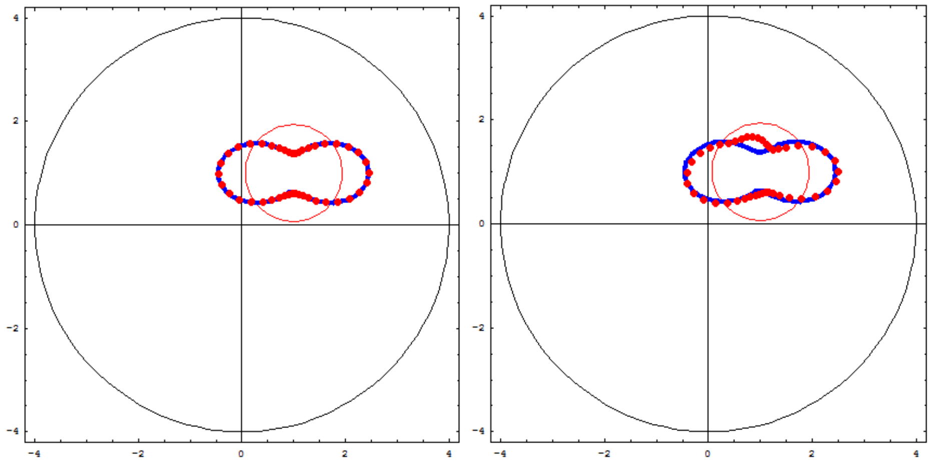


Figure 6. Reconstruction of a kite from partial data without noise (left) and with 5 % of noise (right).

7. Conclusions

In this work, we justified the MFS approximation for direct transmission problems in a potential problem and applied the method to the reconstruction of obstacles in the context of an inverse source problem. We applied two (iterative) methods: one, based on the Kirsh-Kress method which relies on the MFS for Cauchy data reconstruction and the other which is based on the domain derivative (relies on the MFS for transmission problems). The main advantage of the MFS for both approaches is that the method is fast, easy to implement and produces good results, making it well suited for iterative methods. The numerical results shows the feasibility of the reconstruction for both full and partial data. In the last case, we observed some shadow effects on the numerical results.



(a) (b)

Figure 7. Peanut reconstruction in $\mathcal{A}_4$ from full data without noise (left) and with 5 % of noise (right).

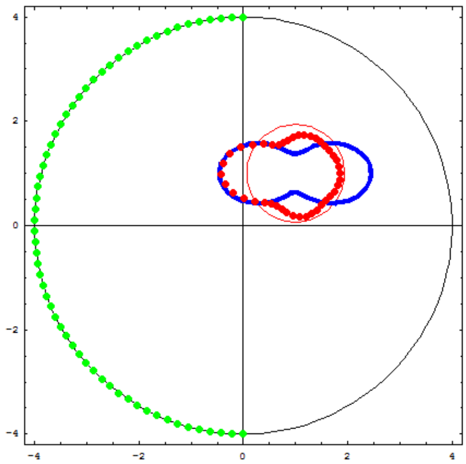


Figure 8. Peanut reconstruction in $\mathcal{A}_4$ (partial data without noise).

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