Exact and near-exact distributions for likelihood ratio test statistics used to test for stationarity and circularity in multivariate models

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In this paper we obtain the exact distribution for the likelihood ratio test (l.r.t.) statistics to test that in a multivariate normal model: i) the mean vector is null and the covariance matrix is circular, ii) the means in the mean vector are all equal and the covariance matrix is circular. The authors show that in the first case the exact distribution of the negative logarithm of the l.r.t. statistic may be written as an infinite mixture of Generalized Near-Integer Gamma (GNIG) distributions, while in the second case it is a Generalized Integer Gamma (GIG) distribution. For the first l.r.t. statistic, in which case the exact distribution is less manageable, it is thus desirable and useful the development of near-exact distributions. Quite extensive numerical studies and simulations show the very high closeness of these near-exact distributions to the exact distribution as well as their very good asymptotic properties.

Keywords: Circular covariance matrix; sums of independent Gamma random variables; sums of independent Logbeta r.v.'s; product of independent Beta random variables

AMS Subject Classification: 62H05, 62H10, 62H15, 62E15, 62E20, 62F03

1. Introduction

Olkin and Press derive in [1] the l.r.t. (likelihood ratio test) statistics to test for circularity of the covariance matrix in a multivariate normal distribution, as well as the statistics to test simultaneously for stationarity (of the mean vector) and circularity of the covariance matrix. We will focus on these latter tests. Olkin and Press in [1] also obtain the expression of the exact distributions of those statistics as products of independent Beta r.v.'s (random variables).

We will use these formulations as the basis for our work and we will show how, working on the c.f. (characteristic function) of the negative logarithm of the l.r.t. statistic, it is possible to obtain quite simple and manageable closed form expressions for both the p.d.f. (probability density function) and c.d.f. (cumulative distribution function) of the statistic to test simultaneously for equal means and circularity of the covariance matrix. This is done in subsection 2.2. Using a similar procedure, in subsection 2.1, we will also show how the exact distribution of the

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l.r.t. statistic to test simultaneously the nullity of all means and circularity of the covariance matrix may be expressed as an infinite mixture of exponentials of GNIG distributions (see [2] and Appendix A.2 for the GNIG distribution). Then in order to avoid the difficulties that might arise in practice from the need to truncate these distributions, we develop in section 3, what we call near-exact distributions for this l.r.t. statistic. These are asymptotic distributions which besides being asymptotic for increasing sample sizes, are also asymptotic for increasing number of variables, laying very close to the exact distribution and matching, by construction, some of the first exact moments. These near-exact distributions are far more manageable than the exact distributions and quite easy to implement computationally, allowing for the easy computation of near-exact quantiles and p-values, which being so close to the exact ones, may, in practice, be used instead of these latter ones. Then in section 4 we show the results of some numerical studies and simulations, which show the very good performance of the near-exact distributions developed.

2. Exact Distributions of test statistics

Let $\mathbf{X} = [X_1, ..., X_p]' \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}_c)$, where $\boldsymbol{\Sigma}_c$ is circular, that is,

$$\boldsymbol{\Sigma}_c = [\sigma_{ij}], \; i,j = 1, ..., p \tag{1}$$

where

$$\sigma_{ii} = Var(X_i) = \sigma_0^2, \qquad \sigma_{i,i+k} = \sigma_{i+k,i} = Cov(X_i, X_{i+k}) = \sigma_0^2 \rho_k, \tag{2}$$

with

$$\rho_k = \rho_{p-k} = Corr(X_i, X_{i+k}), \tag{3}$$

for $i = 1, ..., p$; $k = 1, ..., p - i$. For example, for $p = 6$ and $p = 7$ we have

$$\boldsymbol{\Sigma}_c = \sigma_0^2 \begin{bmatrix} 1 & \rho_1 & \rho_2 & \rho_3 & \rho_2 & \rho_1 \\ \rho_1 & 1 & \rho_1 & \rho_2 & \rho_3 & \rho_2 \\ \rho_2 & \rho_1 & 1 & \rho_1 & \rho_2 & \rho_3 \\ \rho_3 & \rho_2 & \rho_1 & 1 & \rho_1 & \rho_2 \\ \rho_2 & \rho_3 & \rho_2 & \rho_1 & 1 & \rho_1 \\ \rho_1 & \rho_2 & \rho_3 & \rho_2 & \rho_1 & 1 \end{bmatrix}, \qquad \boldsymbol{\Sigma}_c = \sigma_0^2 \begin{bmatrix} 1 & \rho_1 & \rho_2 & \rho_3 & \rho_3 & \rho_2 & \rho_1 \\ \rho_1 & 1 & \rho_1 & \rho_2 & \rho_3 & \rho_3 & \rho_2 \\ \rho_2 & \rho_1 & 1 & \rho_1 & \rho_2 & \rho_3 & \rho_3 \\ \rho_3 & \rho_2 & \rho_1 & 1 & \rho_1 & \rho_2 & \rho_3 \\ \rho_3 & \rho_3 & \rho_2 & \rho_1 & 1 & \rho_1 & \rho_2 \\ \rho_2 & \rho_3 & \rho_3 & \rho_2 & \rho_1 & 1 & \rho_1 \\ \rho_1 & \rho_2 & \rho_3 & \rho_3 & \rho_2 & \rho_1 & 1 \end{bmatrix}.$$

These covariance matrices have the property of being symmetric and cyclic.

2.1. *Simultaneous test that means are zero and covariance matrix is circular*

To test the null hypothesis

$$H_0 : \boldsymbol{\mu} = \mathbf{0}, \; \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_c, \tag{4}$$

based on a sample of size $n+1$, the power $2/(n+1)$ of the l.r.t. statistic may be written [1]

$$\Lambda_1 = 2^{2(p-m-1)}|\mathbf{R}|\prod_{j=1}^{p}\frac{v_{jj}}{v_j+w_j}\,, \tag{5}$$

where $\mathbf{R}$ is the sample correlation matrix and $m=\left\lfloor \frac{p}{2}\right\rfloor$, with $\lfloor\cdot\rfloor$ denoting the floor of the argument, that is, the largest integer that does not exceed the argument. The v_j and w_j are given by (A1) through (A4) and v_{jj} is the j-th diagonal element of the matrix $\mathbf{V}$, in Appendix A.

According to Olkin and Press [1], the l.r.t. statistic (5) has the same distribution as $\prod_{j=1}^{p}U_j$, where the U_j's are independently distributed with

$$U_j \sim \begin{cases} Beta\left(\frac{n-j+1}{2},\frac{j}{2}\right), & j=1,\dots,m+1;\\ Beta\left(\frac{n-j+1}{2},\frac{j+1}{2}\right), & j=m+2,\dots,p\,. \end{cases} \tag{6}$$

The c.f. of $W_1=-log\Lambda_1$ may thus be written as

$$\Phi_{W_1}(t)=\prod_{j=1}^{m+1}\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n-j+1}{2}-it\right)}{\Gamma\left(\frac{n-j+1}{2}\right)\Gamma\left(\frac{n+1}{2}-it\right)}\times\prod_{j=m+2}^{p}\frac{\Gamma\left(\frac{n+2}{2}\right)\Gamma\left(\frac{n-j+1}{2}-it\right)}{\Gamma\left(\frac{n-j+1}{2}\right)\Gamma\left(\frac{n+2}{2}-it\right)}\,. \tag{7}$$

When p is even we have $m+1$ terms in the first product and $m-1$ terms in the second product. In order to have the same number of terms in both products we write

$$\begin{aligned}\Phi_{W_1}(t)=&\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n}{2}-it\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+1}{2}-it\right)}\times\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n-1}{2}-it\right)}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n+1}{2}-it\right)}\\ &\times\underbrace{\prod_{j=3}^{m+1}\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n-j+1}{2}-it\right)}{\Gamma\left(\frac{n-j+1}{2}\right)\Gamma\left(\frac{n+1}{2}-it\right)}}_{m-1\ terms}\times\underbrace{\prod_{j=m+2}^{p}\frac{\Gamma\left(\frac{n+2}{2}\right)\Gamma\left(\frac{n-j+1}{2}-it\right)}{\Gamma\left(\frac{n-j+1}{2}\right)\Gamma\left(\frac{n+2}{2}-it\right)}}_{m-1\ terms}\,,\end{aligned}$$

where the two last products vanish for $p<4$.

Using $\Gamma(r+1)=r\Gamma(r)$, the second term in $\Phi_{W_1}(t)$ above may be written as

$$\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n-1}{2}-it\right)}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n+1}{2}-it\right)}=\frac{\Gamma\left(\frac{n-1}{2}+1\right)\Gamma\left(\frac{n-1}{2}-it\right)}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n-1}{2}-it+1\right)}=\frac{n-1}{2}\left(\frac{n-1}{2}-it\right)^{-1}\,,$$

while the two products may be jointly written as

$$\prod_{\substack{j=3\\ \text{step }2}}^{p-1}\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n+2}{2}\right)\Gamma\left(\frac{n-j+1}{2}-it\right)\Gamma\left(\frac{n-j}{2}-it\right)}{\Gamma\left(\frac{n-j+1}{2}\right)\Gamma\left(\frac{n-j}{2}\right)\Gamma\left(\frac{n+1}{2}-it\right)\Gamma\left(\frac{n+2}{2}-it\right)}\qquad(p\geq 4)\,. \tag{8}$$

But then, using the duplication formula for the Gamma function

$$\Gamma(z)\,\Gamma\left(z+\tfrac{1}{2}\right)=\pi^{\frac{1}{2}}\,2^{1-2z}\Gamma(2z)$$

we may write (8) as

$$\prod_{\substack{j=3\\ step\ 2}}^{p-1} \frac{\Gamma(n+1)\,\Gamma(n-j-2it)}{\Gamma(n-j)\,\Gamma(n+1-2it)} \qquad (p \geq 4)\,,$$

which, using

$$\frac{\Gamma(a+n)}{\Gamma(a)} = \prod_{i=0}^{n-1} (a+i), \quad a \in \ \ , \ n \in \ \ ,$$

may finally be written as

$$\prod_{\substack{j=3\\ step\ 2}}^{p-1} \prod_{k=0}^{j} \left(\frac{n-j+k}{2}\right) \left(\frac{n-j+k}{2} - it\right)^{-1} \qquad (p \geq 4)\,.$$

Taking any product with an upper limit smaller value than the lower limit as evaluating to 1, we may thus write,

$$\Phi_{W_1}(t) = \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n}{2}-it\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+1}{2}-it\right)} \times \frac{n-1}{2}\left(\frac{n-1}{2}-it\right)^{-1} \times \prod_{\substack{j=3\\ step\ 2}}^{p-1} \prod_{k=0}^{j} \left(\frac{n-j+k}{2}\right)\left(\frac{n-j+k}{2}-it\right)^{-1},$$

or

$$\Phi_{W_1}(t) = \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n}{2}-it\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+1}{2}-it\right)} \times \prod_{j=0}^{p-1} \left(\frac{n-j}{2}\right)^{r_j} \left(\frac{n-j}{2}-it\right)^{-r_j}, \tag{9}$$

where

$$r_j = \begin{cases} \frac{p-2}{2}, & j\textit{=0};\\ \frac{p-2}{2}+1, & j\textit{=1};\\ \frac{p-2}{2} - \left\lfloor \frac{|j-2|}{2} \right\rfloor, & j\textit{=2,...,p-1}\,. \end{cases} \tag{10}$$

This shows that the exact distribution of W_1 is, for even p, the distribution of the sum of p independent Gamma r.v.'s with shape parameters $r_j \in$, given by (10), and rate parameters $\frac{n-j}{2}$ $(j = 0, \ldots, p-1)$, which is a GIG distribution with depth p (see [3] and Appendix A.2 for the GIG distribution), plus an independent Logbeta r.v. with parameters $\frac{n}{2}$ and $\frac{1}{2}$.

When p is odd we have $m+1$ terms in the first product in (7) and m terms in the second product, so that we may write

$$\Phi_{W_1}(t)=\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n}{2}-it\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+1}{2}-it\right)} \times\underbrace{\prod_{j=2}^{m+1}\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n-j+1}{2}-it\right)}{\Gamma\left(\frac{n-j+1}{2}\right)\Gamma\left(\frac{n+1}{2}-it\right)}}_{m\ terms}\times\underbrace{\prod_{j=m+2}^{p}\frac{\Gamma\left(\frac{n+2}{2}\right)\Gamma\left(\frac{n-j+1}{2}-it\right)}{\Gamma\left(\frac{n-j+1}{2}\right)\Gamma\left(\frac{n+2}{2}-it\right)}}_{m\ terms}.$$

Following a similar procedure to the one used in the case where p was even, we obtain

$$\Phi_{W_1}(t)=\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n}{2}-it\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+1}{2}-it\right)}\times\prod_{\substack{j=2\\ step\ 2}}^{p-1}\prod_{k=0}^{j}\left(\frac{n-j+k}{2}\right)\left(\frac{n-j+k}{2}-it\right)^{-1}$$

or

$$\Phi_{W_1}(t)=\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n}{2}-it\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+1}{2}-it\right)}\times\prod_{j=0}^{p-1}\left(\frac{n-j}{2}\right)^{r_j}\left(\frac{n-j}{2}-it\right)^{-r_j} \tag{11}$$

with

$$r_j=\frac{p-1}{2}-\left\lfloor\frac{|j-1|}{2}\right\rfloor,\ j=0,...,p-1. \tag{12}$$

The exact distribution of W_1, is thus, for odd p, the sum of p independent Gamma random variables with shape parameters $r_j\in$, given by (12), and rate parameters $\frac{n-j}{2}$ $(j=0,\dots,p-1)$, which is a GIG r.v. with depth p, plus an independent Logbeta r.v. with parameters $\frac{n}{2}$ and $\frac{1}{2}$.

Using the results in Appendix B.1 we may thus write the c.f. of W_1 as

$$\begin{aligned}\Phi_{W_1}(t)&=\left\{\sum_{k=0}^{\infty}p_k^*\left(\frac{n}{2}\right)^{\frac{1}{2}+k}\left(\frac{n}{2}-it\right)^{-\left(\frac{1}{2}+k\right)}\right\}\times\left\{\prod_{j=0}^{p-1}\left(\frac{n-j}{2}\right)^{r_j}\left(\frac{n-j}{2}-it\right)^{-r_j}\right\}\\&=\sum_{k=0}^{\infty}p_k^*\prod_{j=0}^{p-1}\left(\frac{n-j}{2}\right)^{r_j^*}\left(\frac{n-j}{2}-it\right)^{-r_j^*}\end{aligned} \tag{13}$$

with

$$r_j^*=\begin{cases}r_0+\frac{1}{2}+k & j=0\\ r_j & j=1,\dots,p-1\,,\end{cases} \tag{14}$$

where $r_0,\dots,r_{p-1}$ are given by (10) for even p and (12) for odd p, and $p_k^*=\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\frac{p_k\left(\frac{1}{2}\right)}{\left(\frac{n}{2}\right)^{\frac{1}{2}+k}}$, where $p_k\left(\frac{1}{2}\right)$ is given by (B3) and (B4) in Appendix B.1, with $a=\frac{n}{2}$ and $b=\frac{1}{2}$.

This shows that the exact distribution of W_1 is an infinite mixture of GNIG distributions of depth p, with weights p_k^*, with p.d.f. and c.d.f. given by

$$f_{W_1}(w) = \sum_{k=0}^{\infty} p_k^* \, f^{GNIG}\left(w \,\middle|\, r_1, \ldots, r_{p-1}; r_0 + \frac{1}{2} + k; \frac{n-1}{2}, \ldots, \frac{n-(p-1)}{2}; \frac{n}{2}; p\right)$$

and

$$F_{W_1}(w) = \sum_{k=0}^{\infty} p_k^* \, F^{GNIG}\left(w \,\middle|\, r_1, \ldots, r_{p-1}; r_0 + \frac{1}{2} + k; \frac{n-1}{2}, \ldots, \frac{n-(p-1)}{2}; \frac{n}{2}; p\right).$$

Thus the exact p.d.f. and c.d.f. of Λ_1 are

$$f_{\Lambda_1}(\ell) = \sum_{k=0}^{\infty} p_k^* \, f^{GNIG}\left(-\log \ell \,\middle|\, r_1, \ldots, r_{p-1}; r_0 + \frac{1}{2} + k; \frac{n-1}{2}, \ldots, \frac{n-(p-1)}{2}; \frac{n}{2}; p\right) \frac{1}{\ell}$$

and

$$F_{\Lambda_1}(\ell) = \sum_{k=0}^{\infty} p_k^* \left\{1 - F^{GNIG}\left(-\log \ell \,\middle|\, r_1, \ldots, r_{p-1}; r_0 + \frac{1}{2} + k; \frac{n-1}{2}, \ldots, \frac{n-(p-1)}{2}; \frac{n}{2}; p\right)\right\}.$$

2.2. *Simultaneous test that means are equal and covariance matrix is circular*

Let

$$\boldsymbol{\mu} = [\mu_1, \ldots, \mu_p]' \,.$$

To test the null hypothesis

$$H_0 : \mu_1 = \ldots = \mu_p; \;\; \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_c \,,$$

based on a sample of size $n+1$, the power $2/(n+1)$ of the l.r.t. statistic may be written [1],

$$\Lambda_2 = 2^{2(p-m-1)} |R| \prod_{j=2}^{p} \frac{v_{jj}}{v_j + w_j} \,, \tag{15}$$

where, as in (5), the v_j and w_j are given by (A1) through (A4) and v_{jj} is the j-th diagonal element of the matrix $\mathbf{V}$, in Appendix A (we may note that there is a small typo in the original paper where v_{jj} is written v_j).

The statistic Λ_2 in (15) has the same distribution of $\prod_{j=2}^{p} U_j$, where the U_j's are the same as in (6).

Following the development of section (2.1) we obtain for even p

$$\Phi_{W_2}(t) = \prod_{j=0}^{p-1} \left(\frac{n-j}{2}\right)^{r_j} \left(\frac{n-j}{2} - it\right)^{-r_j},$$

with the r_j given by (10). The exact distribution of W_2, for even p, is thus the same as the distribution of the sum of p independent Gamma r.v.'s with shape parameters $r_j \in$, given by (10), and rate parameters $\frac{n-j}{2}$ $(j = 0, \ldots, p-1)$, which is a GIG distribution with depth p, with the above shape and rate parameters.

For odd p we have the r_j given by (12). This shows that in this case, the exact distribution of W_2 is the same as the distribution of the sum of p independent Gamma random variables with shape parameters $r_j \in$, given by (12), and rate parameters $\frac{n-j}{2}$ $(j = 0, \ldots, p-1)$, which is once again a GIG distribution with depth p, with the above shape and rate parameters.

Using the notation in Appendix B, the exact p.d.f. and c.d.f. of W_2 and Λ_2 are thus

$$f_{W_2}(w) = f^{GIG}\left(w|r_0, \ldots, r_{p-1}; \frac{n}{2}, \ldots, \frac{n-p+1}{2}; p\right),$$

$$F_{W_2}(w) = F^{GIG}\left(w|r_0, \ldots, r_{p-1}; \frac{n}{2}, \ldots, \frac{n-p+1}{2}; p\right)$$

and

$$f_{\Lambda_2}(\ell) = f^{GIG}\left(-\log\, \ell|r_0, \ldots, r_{p-1}; \frac{n}{2}, \ldots, \frac{n-p+1}{2}; p\right)\frac{1}{\ell}, \tag{16}$$

$$F_{\Lambda_2}(\ell) = 1 - F^{GIG}\left(-\log\, \ell|r_0, \ldots, r_{p-1}; \frac{n}{2}, \ldots, \frac{n-p+1}{2}; p\right), \tag{17}$$

with $r_0, \ldots, r_{p-1}$ given by (10) for even p and by (12) for odd p, and where w and ℓ denote respectively the running values of W_2 and Λ_2.

These results match the ones in [5]. They also allow for a very efficient and simple computation of quantiles. Some quite extensive tables of exact quantiles are shown in Appendix C. Computing times for these quantiles vary from less than a millisecond to about 15 or 16 milliseconds for p ranging from 3 to 9 and from 15 or 16 milliseconds to around 30 milliseconds for the larger sample sizes for $p = 9$ and $p = 10$. For $p = 14$ quantiles are obtained respectively in less than 2 tenths of a second and for $p = 16$ in around 3 tenths of a second. For $p = 20$ and 25 in little over half of a second, and in less than a second to about a second for $p = 30$, and in 6-8.5 seconds for $p = 50$. These computing times were obtained with version 5.2 of Mathematica as the only software running on an Intel P8700 processor running at 2.53 GHz, with 4GB of RAM, running the Windows 7 Home Premium 64-bit operating system.

In Appendix D are provided Mathematica modules to compute the exact p.d.f. and c.d.f. of Λ_2. With the module used to compute the c.d.f. we may also easily compute the exact quantiles for Λ_2. An example of a command that may be used for such computation is also provided in Appendix D.

In Figure 1 we may observe a few plots of exact p.d.f.'s and c.d.f.'s of Λ_2, obtained with the modules in Appendix D.

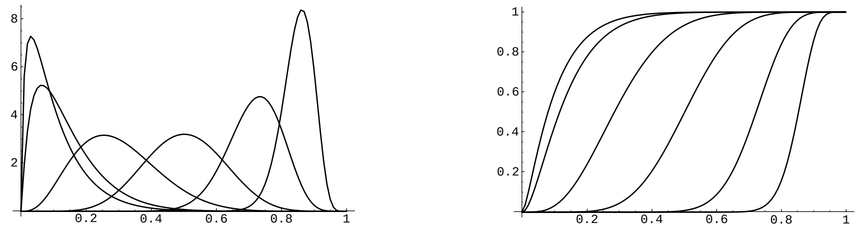

Figure 1 – Plots of exact p.d.f.'s (on the left) and c.d.f.'s (on the right) of Λ_2 for $p = 5$ and sample sizes of 9, 10, 15, 25, 50 and 100 for modes of p.d.f.'s and c.d.f.'s ranging from left to right.

3. Near-exact distributions for the l.r.t. statistics in subsection 2.1

From (13) in subsection 2.1, we will take

$$\Phi^*_{W_1}(t) = \sum_{k=0}^{s} p_k^{**} \prod_{j=0}^{p-1} \left(\frac{n-j}{2}\right)^{r_j^*} \left(\frac{n-j}{2} - it\right)^{-r_j^*} \tag{18}$$

as near-exact c.f. of W_1, where the p_k^{**}, $k = 0, \ldots, s-1$, are obtained as the solution of the system of s equations

$$\left.\frac{\partial^{k+1}}{\partial t^{k+1}} \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n}{2} - it\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+1}{2} - it\right)}\right|_{t=0} = \left.\frac{\partial^{k+1}}{\partial t^{k+1}} \sum_{k=0}^{s} p_k^{**} \left(\frac{n}{2}\right)^{\frac{1}{2}+k} \left(\frac{n}{2} - it\right)^{-\left(\frac{1}{2}+k\right)}\right|_{t=0},$$
$$k = 0, \ldots, s-1\,,$$

and

$$p_s^{**} = 1 - \sum_{k=0}^{s-1} p_k^{**}\,.$$

This way, $\Phi^*_{W_1}(t)$ in (18) will equate the first s derivatives of $\Phi_{W_1}(t)$, that is, the exact c.f. of W_1, at $t = 0$, or, in other words, the near-exact distribution corresponding to $\Phi^*_{W_1}(t)$ in (18) will equate the first s exact moments of W_1. This distribution will be a finite mixture of $s+1$ GNIG distributions of depth p, each one of them with rate parameters $\frac{n-j}{2}$ $(j = 0, \ldots, p-1)$ and shape parameters $r^*_{j,k}$ given by (14).

The near-exact distributions built using the above procedure, will yield for W_1 near-exact distributions which are finite mixtures of $s+1$ GNIG distributions of depth p, with p.d.f.'s and c.d.f.'s respectively of the form

$$f_{W_1}(w) = \sum_{k=0}^{s} p_k^{**} f^{GNIG}\!\left(w \,\middle|\, r_1, \ldots, r_{p-1}; r_0 + \frac{1}{2} + k; \frac{n-1}{2}, \ldots, \frac{n-(p-1)}{2}; \frac{n}{2}; p\right)$$

and

$$F_{W_1}(w)=\sum_{k=0}^{s}p_k^{**}F^{GNIG}\left(w\,\middle|\,r_1,\dots,r_{p-1};r_0+\frac{1}{2}+k;\frac{n-1}{2},\dots,\frac{n-(p-1)}{2};\frac{n}{2};p\right),$$

and for Λ_1, p.d.f.'s and c.d.f.'s of the form

$$f_{\Lambda_1}(\ell)=\sum_{k=0}^{s}p_k^{**}f^{GNIG}\left(-\log\,\ell\,\middle|\,r_1,\dots,r_{p-1};r_0+\frac{1}{2}+k;\right.$$
$$\left.\frac{n-1}{2},\dots,\frac{n-(p-1)}{2}2;\frac{n}{2};p\right)\frac{1}{\ell}$$

and

$$F_{\Lambda_1}(\ell)=\sum_{k=0}^{s}p_k^{**}\left\{1-F^{GNIG}\left(-\log\,\ell\,\middle|\,r_1,\dots,r_{p-1};r_0+\frac{1}{2}+k;\right.\right.$$
$$\left.\left.\frac{n-1}{2},\dots,\frac{n-(p-1)}{2};\frac{n}{2};p\right)\right\}.$$

In the next section some numerical studies are performed in order to assess the performance and the closeness of these near-exact distributions to the exact distribution.

4. Simulations and numerical studies

Since in the multivariate setting we are working the alternatives which might be considered to the null hypothesis in (4) are so vast and also since our aim is not to compare the performance of different tests neither to assess the performance of the test under study, for whose statistic we developed near-exact distributions, but rather to assess the performance of these near-exact distributions, we will not undertake any power studies but we will rather carry out some numerical studies and simulations in order to have a fine assessment of the performance and advantages of the near-exact distributions developed in this paper for the statistic Λ_1 in (5). Anyway, we may note that the distributions in [6] may be seen as non-null distributions for Λ_1, under a broad alternative to H_0 in (4) which would be stated as

$$H_1:\,\boldsymbol{\Sigma}=\boldsymbol{\Sigma}_c\;\;\text{(and any }\boldsymbol{\mu}\in\ ^{p}\text{)}\,.$$

4.1. *Asymptotic distribution for* $\boldsymbol{W_1=-\log\Lambda_1}$

We will use the Box-style asymptotic distribution in Section 5.1 of [1] as a comparison basis for the near-exact distributions developed for Λ_1. For $W_1=-\log\Lambda_1$ this asymptotic distribution yields, for a sample of size $N=n+1$,

$$\begin{aligned}P\left(-2\rho\log\,\Lambda_1^{N/2}\le z\right)&=P\left(N\rho\,W_1\le z\right)\\&=(1+w)\,P\left(\chi^2_f\le z\right)+w\,P\left(\chi^2_{f+4}\le z\right)+O(N^{-3})\,,\end{aligned}\tag{19}$$

so that it yields for W_1 the Box-style asymptotic c.f.

$$\Phi^{Box}_{W_1}(N\rho t)=(1+w)\left(\frac{1}{2}\right)^{f/2}\left(\frac{1}{2}-it\right)^{-f/2}-w\left(\frac{1}{2}\right)^{2+f/2}\left(\frac{1}{2}-it\right)^{-2-f/2}$$

or

$$\begin{aligned}\Phi^{Box}_{W_1}(t)&=(1+w)\left(\tfrac{1}{2}\right)^{f/2}\left(\tfrac{1}{2}-i\tfrac{t}{N\rho}\right)^{-f/2}-w\left(\tfrac{1}{2}\right)^{2+f/2}\left(\tfrac{1}{2}-i\tfrac{t}{N\rho}\right)^{-2-f/2}\\&=(1+w)\left(\tfrac{N\rho}{2}\right)^{f/2}\left(\tfrac{N\rho}{2}-it\right)^{-f/2}-w\left(\tfrac{N\rho}{2}\right)^{2+f/2}\left(\tfrac{N\rho}{2}-it\right)^{-2-f/2}\end{aligned}\tag{20}$$

which is the c.f. of a mixture of two Gamma distributions, the first one with weight $1+w$ and shape parameter equal to $f/2$ and the second one with weight w and shape parameter $2+f/2$, both with rate parameter $N\rho/2$. Concerning the expressions for the parameters ρ, f and w we will derive them from the foundational expressions in order to be able to deal with some typos in [1]. In (20), we have

$$f=2\sum_{j=1}^{p}(\eta_j-\xi_j)=p-m-1+\frac{p(p+1)}{2}\,,\quad m=\lfloor p/2\rfloor\,,$$

$$\rho=1-\frac{2b}{N}$$

and

$$\begin{aligned}w&=\frac{1}{6\left(\frac{N}{2}-b\right)^2}\sum_{j=1}^{p}\left\{B_3(b+\xi_j)-B_3(b+\eta_j)\right\}\\&=\frac{1}{6\left(\frac{N}{2}-b\right)^2}\left[\sum_{j=1}^{p}\left\{B_3\left(b-\tfrac{j}{2}\right)-B_3(b)\right\}-(p-m-1)\left(B_3\left(b+\tfrac{1}{2}\right)-B_3(b)\right)\right]\\&=a_0+a_1b+a_2b^2\end{aligned}$$

with

$$\eta_1=\cdots=\eta_{m+1}=0\,,\qquad \eta_{m+2}=\cdots=\eta_p=\frac{1}{2}\,,$$

$$\xi_j=-\frac{j}{2}\,,\quad j=1,\ldots,p\,,$$

$$b=\frac{1}{2}+\frac{1}{f}\sum_{j=1}^{p}\left(\xi_j^2-\eta_j^2\right)=\frac{1}{24f}\left\{p(p+1)(2p+7)+6(p-m-1)\right\}\,,$$

$$B_3(z)=\frac{1}{2}\left(2z^3-3z^2+z\right)$$

and

$$\begin{aligned} a_0 &= -\tfrac{3}{16}p(p+1) - \tfrac{5}{32}p^2(p+1) - \tfrac{1}{32}p^3(p+1) \\ &= -p(p+1)(p^2+5p+6)\tfrac{1}{32} \\ a_1 &= \tfrac{7}{8}p(p+1) + \tfrac{1}{4}p^2(p+1) + \tfrac{3}{4}(p-m-1) = 3bf \\ a_2 &= -\tfrac{3}{4}p(p+1) - \tfrac{3}{2}(p-m-1) = -\tfrac{3}{2}f\,, \end{aligned}$$

so that we may write

$$\begin{aligned} w &= a_0 + 3b^2 f - \tfrac{3}{2}fb^2 \\ &= a_0 + \tfrac{3}{2}b^2 f \\ &= -p(p+1)(p^2+5p+6)/32 + 3b^2 f/2\,. \end{aligned}$$

However, as we will see in the next subsection, the bound in (19) is usually not met even for quite large sample sizes, and the situation worsens when the dimension (p) increases.

4.2. *A proximity measure between distributions based on c.f.'s*

In order to better address and study the proximity between distributions in cases where we have both c.f.'s available but at least one of the c.d.f.'s is not available or at least not available in a precise and easily computable form, as it happens when comparing the exact distribution with a near-exact or asymptotic distribution, we will use the measure

$$\Delta = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|\frac{\Phi_W(t)-\Phi^*_W(t)}{t}\right| dt \tag{21}$$

with

$$\max_w |F_W(w) - F^*_W(w)|\;, \tag{22}$$

where $\Phi_W(t)$ represents the exact c.f. of W and $\Phi^*_W(t)$ the asymptotic or near-exact c.f., and $F_W(\,\cdot\,)$ represents the exact c.d.f. of W and $F^*_W(\,\cdot\,)$ the c.d.f. corresponding to $\Phi^*_W(t)$.

From both (21) and (22) it is clear that better approximations to the exact distribution will yield smaller values of Δ.

The values of Δ in Table 1 were computed using (7) for the exact c.f. of $W_1 = -\log\Lambda_1$, (20) for the asymptotic and (18), with $s = 2, 4, 6, 10$, for the near-exact c.f..

We may see how the values of Δ in Table 1 show that the near-exact distributions besides having a much better performance than the asymptotic distribution they are indeed asymptotic not only for increasing sample size but also for increasing number of variables, of course with the ones that match a larger number of exact moments displaying a much better performance. The asymptotic character of the near-exact distributions seems to be even more accentuated for larger values of s, that is, for larger values of the number of exact moments matched.

Table 1 – Values of the measure Δ for the asymptotic and near-exact distributions for the statistic Λ_1, for different values of p and N.

p	N	Box	Near-exact distributions number of exact moments equated			
			2	4	6	10
5	5	1.22×10^{-1}	8.12×10^{-7}	1.07×10^{-9}	6.76×10^{-11}	1.25×10^{-13}
	10	3.42×10^{-3}	3.58×10^{-7}	6.66×10^{-10}	1.20×10^{-12}	3.28×10^{-15}
	50	1.18×10^{-5}	4.36×10^{-9}	5.47×10^{-13}	2.34×10^{-16}	3.24×10^{-22}
	100	1.35×10^{-6}	5.63×10^{-10}	1.81×10^{-14}	2.02×10^{-18}	2.26×10^{-25}
10	10	3.49×10^{-1}	9.41×10^{-9}	2.22×10^{-12}	4.13×10^{-16}	8.61×10^{-20}
	50	2.26×10^{-4}	6.32×10^{-10}	2.58×10^{-14}	4.00×10^{-18}	9.17×10^{-25}
	100	2.28×10^{-5}	8.71×10^{-11}	9.52×10^{-16}	4.02×10^{-20}	8.08×10^{-28}
	200	2.59×10^{-6}	1.14×10^{-11}	3.20×10^{-17}	3.48×10^{-22}	4.82×10^{-31}
15	15	5.78×10^{-1}	6.67×10^{-10}	3.20×10^{-14}	4.55×10^{-18}	5.60×10^{-24}
	50	1.45×10^{-3}	1.78×10^{-10}	3.25×10^{-15}	2.31×10^{-19}	1.21×10^{-26}
	100	1.25×10^{-4}	2.64×10^{-11}	1.35×10^{-16}	2.77×10^{-21}	1.41×10^{-29}
	200	1.32×10^{-5}	3.55×10^{-12}	4.79×10^{-18}	2.58×10^{-23}	9.44×10^{-33}
20	20	7.89×10^{-1}	1.03×10^{-10}	1.50×10^{-15}	7.94×10^{-20}	2.60×10^{-27}
	50	6.21×10^{-3}	6.82×10^{-11}	6.65×10^{-16}	2.56×10^{-20}	4.00×10^{-28}
	100	4.39×10^{-4}	1.10×10^{-11}	3.21×10^{-17}	3.76×10^{-22}	6.49×10^{-31}
	200	4.26×10^{-5}	1.53×10^{-12}	1.20×10^{-18}	3.78×10^{-24}	4.93×10^{-34}

4.3. *Simulation studies and simulation-related results*

Although once obtained the results in Table 1 there is indeed not much more left to be looked upon, we still undertook a quite extensive simulation study to try to better assess some fine details pertaining to the performance of the near-exact distributions, mainly in comparison with the asymptotic distribution in [1].

For each combination of values of p and N in Table 1 we obtained a pseudo-random sample of size 2×10^6 from the distribution of Λ_1, generated directly from the distribution of the r.v.'s U_j $(j = 1, \ldots, p)$ in (6).

Then, on the left hand side of Figures 2-5 we have, for the two smaller sample sizes considered for each value of p in Table 1, the superimposed plots of the relative frequency histograms of the pseudo-random sample and the p.d.f.'s of the asymptotic and the near-exact distribution for $s = 2$, that is, the near-exact distribution that matches only two exact moments, while on the right hand side of the same Figures we have the superimposed plots of the cumulative frequency histograms of the simulated data and the c.d.f.'s of the asymptotic and near-exact distribution for $s = 2$.

In Figures 2-5 we may see, for all cases, the very good agreement between the histograms from the simulated data and both the p.d.f's and c.d.f.'s for the near-exact distributions that match 2 exact moments, while for the smaller sample sizes the agreement of the asymptotic p.d.f. and c.d.f. is not good at all, with a clear worsening of the approximation, for a given sample size, when the number of variables involved (p) increases. Actually on some of the plots the asymptotic p.d.f. has so low values that this p.d.f. is almost not visible.

Although for the larger sample sizes the agreement of the plots of the asymptotic distribution may seem to be almost perfect, indeed this is not quite the case, as it may be shown by more detailed studies.

This way, in order to be able to have a more detailed evaluation of the behavior of the asymptotic and near-exact distributions, namely in terms of quantiles, we have decided to compute for each combination of p and n in Table 1 the simulated, the asymptotic and the near-exact 0.05 and 0.01 quantiles and then analyze the value given by the exact c.d.f. for each of these quantiles.

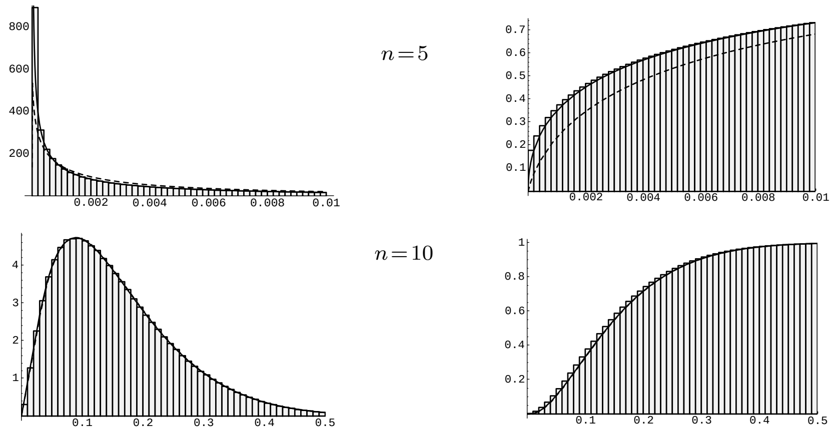

Figure 2 – Plots of the histograms obtained from simulated data and the asymptotic (- - - -) and near-exact (——) p.d.f.'s (on the left) and c.d.f.'s (on the right) for $p = 5$.

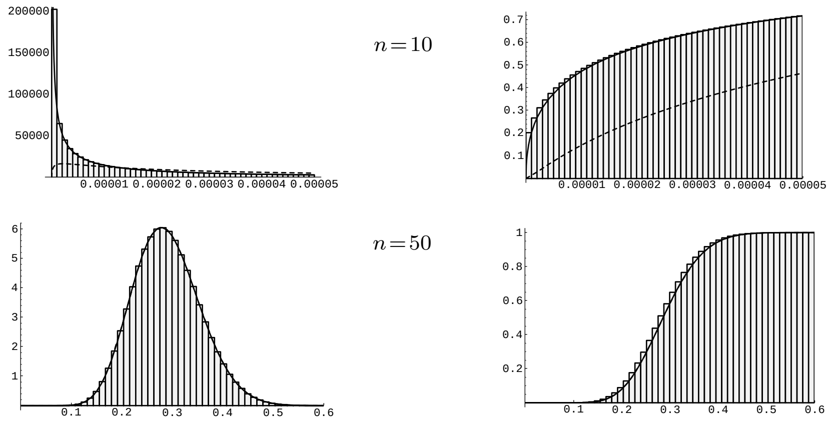

Figure 3 – Plots of the histograms obtained from simulated data and the asymptotic (- - - -) and near-exact (——) p.d.f.'s (on the left) and c.d.f.'s (on the right) for $p = 10$.

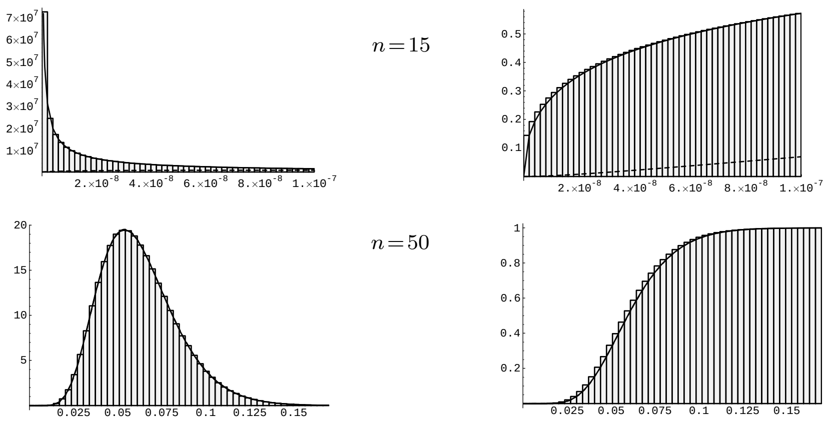

Figure 4 – Plots of the histograms obtained from simulated data and the asymptotic (- - - -) and near-exact (——) p.d.f.'s (on the left) and c.d.f.'s (on the right) for $p = 15$.

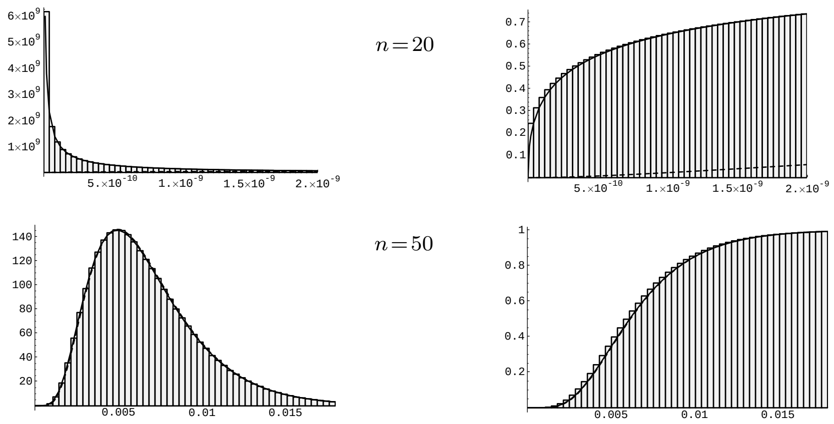

Figure 5 – Plots of the histograms obtained from simulated data and the asymptotic (----) and near-exact (——) p.d.f.'s (on the left) and c.d.f.'s (on the right) for $p = 20$.

The results are summarized in Tables 2 and 3 where we have respectively the values of the differences between 0.05 and 0.01 and the left tail probability values given by the exact c.d.f. on the corresponding quantiles quantiles obtained from the simulated values, the asymptotic distribution in [1] and the near-exact distributions proposed in this paper, for $s = 2, 4, 6$ and 10 exact moments matched.

We may see how the asymptotic distribution performs quite poorly for the smaller sample sizes, with the quantiles from the simulated pseudo-random samples with a much better performance than the asymptotic quantiles in these cases. Anyway, even the quantiles from the near-exact distribution which equates only two exact moments show a remarkably better performance than both the asymptotic and simulation quantiles, so that in cases where some further precision is needed the near-exact distributions may be used with great advantage, moreover since, as we will better see ahead, they even present a good advantage from the point of view of computing times.

In order to be able to achieve the necessary precision the values for the exact c.d.f. of Λ_1 had to be computed through the use of the inversion formula

$$F_{\Lambda_1}(z) = \frac{1}{2} - \frac{1}{\pi} \int_0^{+\infty} Im \left(\frac{z^{it}}{t} \, \Phi_{W_1}(t) \right) dt \tag{23}$$

which is derived from the inversion formulas in [7] and where $W_1 = -\log \Lambda_1$.

Furthermore, in (23), the expression that has to be used for $\Phi_W(t)$ is the one in (9) or (11) and not the one in (7). Anyway, the computation time and the correct handling of precision will always remain as issues that will render not feasible the computation of exact quantiles with the required precision.

In Tables 4-7 are displayed the 0.05 and 0.01 quantiles for Λ_1 for all the combinations of values of p and n used in Table 1. Given that clearly as the number of exact moments matched by the near-exact distributions increases, the near-exact distributions converge to the exact distribution, the quantiles given by the near-exact distributions that match 11 exact moments were used as a benchmark. Then, the values of the quantiles shown in the Tables are rounded till at most one non-conforming digit shows up. That is, if the last digit of a given quantile agrees with more precise quantiles, it means that it was originated from rounding off. For the asymptotic quantiles, in cases in which they do not even match the first significant digit, two non-conformable digits are shown, in order to allow for an easier reading of the value.

Table 2 – Absolute value of the differences between 0.05 and the left tail probability given by the exact distribution for the simulated, asymptotic and near-exact quantiles for Λ_1.

p	n	Simul.	Box	Near-exact distributions number of exact moments equated 2	4	6	10
5	5	$3.14{\times}10^{-5}$	$8.85{\times}10^{-2}$	$1.06{\times}10^{-7}$	$2.09{\times}10^{-11}$	$5.23{\times}10^{-13}$	$6.45{\times}10^{-16}$
	10	$3.08{\times}10^{-5}$	$2.15{\times}10^{-3}$	$4.67{\times}10^{-8}$	$9.49{\times}10^{-11}$	$2.19{\times}10^{-13}$	$7.80{\times}10^{-16}$
	50	$2.98{\times}10^{-5}$	$6.12{\times}10^{-6}$	$1.36{\times}10^{-10}$	$3.42{\times}10^{-14}$	$6.54{\times}10^{-17}$	$5.67{\times}10^{-23}$
	100	$1.02{\times}10^{-4}$	$6.68{\times}10^{-7}$	$1.14{\times}10^{-11}$	$1.54{\times}10^{-15}$	$6.25{\times}10^{-19}$	$2.91{\times}10^{-26}$
10	10	$1.20{\times}10^{-4}$	$2.74{\times}10^{-1}$	$1.57{\times}10^{-9}$	$7.17{\times}10^{-14}$	$8.61{\times}10^{-18}$	$3.08{\times}10^{-22}$
	50	$3.80{\times}10^{-4}$	$1.10{\times}10^{-4}$	$1.49{\times}10^{-10}$	$8.61{\times}10^{-15}$	$1.24{\times}10^{-18}$	$5.21{\times}10^{-26}$
	100	$8.73{\times}10^{-5}$	$9.98{\times}10^{-6}$	$1.97{\times}10^{-11}$	$3.04{\times}10^{-16}$	$1.14{\times}10^{-20}$	$3.44{\times}10^{-28}$
	200	$5.76{\times}10^{-5}$	$1.05{\times}10^{-6}$	$2.52{\times}10^{-12}$	$9.95{\times}10^{-18}$	$9.33{\times}10^{-23}$	$4.18{\times}10^{-32}$
15	15	$3.89{\times}10^{-5}$	$4.83{\times}10^{-1}$	$1.26{\times}10^{-10}$	$1.12{\times}10^{-15}$	$1.54{\times}10^{-19}$	$2.63{\times}10^{-27}$
	50	$1.49{\times}10^{-4}$	$7.40{\times}10^{-4}$	$5.53{\times}10^{-11}$	$1.43{\times}10^{-15}$	$1.04{\times}10^{-19}$	$5.04{\times}10^{-27}$
	100	$2.61{\times}10^{-4}$	$5.61{\times}10^{-5}$	$7.99{\times}10^{-12}$	$6.25{\times}10^{-17}$	$1.27{\times}10^{-21}$	$5.83{\times}10^{-30}$
	200	$1.35{\times}10^{-4}$	$5.28{\times}10^{-6}$	$1.06{\times}10^{-12}$	$2.10{\times}10^{-18}$	$1.19{\times}10^{-23}$	$3.78{\times}10^{-33}$
20	20	$7.59{\times}10^{-5}$	$6.70{\times}10^{-1}$	$2.11{\times}10^{-11}$	$5.80{\times}10^{-17}$	$3.34{\times}10^{-21}$	$2.37{\times}10^{-30}$
	50	$1.40{\times}10^{-4}$	$3.26{\times}10^{-3}$	$2.36{\times}10^{-11}$	$3.14{\times}10^{-16}$	$1.16{\times}10^{-20}$	$1.49{\times}10^{-28}$
	100	$1.81{\times}10^{-4}$	$2.10{\times}10^{-4}$	$3.75{\times}10^{-12}$	$1.54{\times}10^{-17}$	$1.83{\times}10^{-22}$	$2.97{\times}10^{-31}$
	200	$3.54{\times}10^{-5}$	$1.77{\times}10^{-5}$	$5.16{\times}10^{-13}$	$5.78{\times}10^{-19}$	$1.88{\times}10^{-24}$	$2.38{\times}10^{-34}$

Table 3 – Absolute value of the differences between 0.01 and the left tail probability given by the exact distribution for the simulated, asymptotic and near-exact quantiles for Λ_1.

p	n	Simul.	Box	Near-exact distributions number of exact moments equated 2	4	6	10
5	5	$1.18{\times}10^{-4}$	$4.84{\times}10^{-2}$	$3.25{\times}10^{-8}$	$6.42{\times}10^{-12}$	$3.45{\times}10^{-13}$	$8.00{\times}10^{-17}$
	10	$7.61{\times}10^{-6}$	$8.87{\times}10^{-4}$	$6.65{\times}10^{-8}$	$8.88{\times}10^{-11}$	$4.00{\times}10^{-14}$	$2.98{\times}10^{-16}$
	50	$6.07{\times}10^{-5}$	$2.29{\times}10^{-6}$	$9.45{\times}10^{-10}$	$1.16{\times}10^{-13}$	$4.14{\times}10^{-17}$	$1.78{\times}10^{-23}$
	100	$1.02{\times}10^{-4}$	$2.44{\times}10^{-7}$	$1.23{\times}10^{-10}$	$3.88{\times}10^{-15}$	$3.52{\times}10^{-19}$	$1.96{\times}10^{-26}$
10	10	$1.80{\times}10^{-5}$	$1.79{\times}10^{-1}$	$4.62{\times}10^{-10}$	$2.82{\times}10^{-14}$	$4.10{\times}10^{-19}$	$4.66{\times}10^{-23}$
	50	$9.30{\times}10^{-5}$	$3.82{\times}10^{-5}$	$1.68{\times}10^{-10}$	$5.10{\times}10^{-15}$	$5.49{\times}10^{-19}$	$1.27{\times}10^{-25}$
	100	$3.21{\times}10^{-5}$	$3.32{\times}10^{-6}$	$2.36{\times}10^{-11}$	$2.05{\times}10^{-16}$	$6.68{\times}10^{-21}$	$1.49{\times}10^{-28}$
	200	$8.09{\times}10^{-6}$	$3.38{\times}10^{-7}$	$3.11{\times}10^{-12}$	$7.13{\times}10^{-18}$	$6.25{\times}10^{-23}$	$9.87{\times}10^{-32}$
15	15	$3.32{\times}10^{-5}$	$3.60{\times}10^{-1}$	$3.76{\times}10^{-11}$	$5.49{\times}10^{-16}$	$1.56{\times}10^{-20}$	$3.43{\times}10^{-27}$
	50	$1.46{\times}10^{-4}$	$2.55{\times}10^{-4}$	$4.88{\times}10^{-11}$	$4.64{\times}10^{-16}$	$7.77{\times}10^{-21}$	$1.07{\times}10^{-27}$
	100	$1.26{\times}10^{-4}$	$1.85{\times}10^{-5}$	$7.44{\times}10^{-12}$	$2.24{\times}10^{-17}$	$1.27{\times}10^{-22}$	$5.68{\times}10^{-31}$
	200	$2.32{\times}10^{-5}$	$1.66{\times}10^{-6}$	$1.01{\times}10^{-12}$	$8.40{\times}10^{-19}$	$1.58{\times}10^{-24}$	$1.37{\times}10^{-34}$
20	20	$9.59{\times}10^{-5}$	$5.60{\times}10^{-1}$	$6.37{\times}10^{-12}$	$3.07{\times}10^{-17}$	$3.43{\times}10^{-22}$	$2.11{\times}10^{-30}$
	50	$9.40{\times}10^{-5}$	$1.11{\times}10^{-3}$	$1.88{\times}10^{-11}$	$6.95{\times}10^{-17}$	$1.55{\times}10^{-21}$	$7.34{\times}10^{-29}$
	100	$6.68{\times}10^{-5}$	$6.98{\times}10^{-5}$	$3.15{\times}10^{-12}$	$4.15{\times}10^{-18}$	$1.16{\times}10^{-23}$	$1.10{\times}10^{-31}$
	200	$2.09{\times}10^{-4}$	$5.60{\times}10^{-6}$	$4.43{\times}10^{-13}$	$1.68{\times}10^{-19}$	$6.16{\times}10^{-26}$	$7.68{\times}10^{-35}$

We may see how for both quantiles, for a given p, as the sample size increases, the number of correct digits increases for both the asymptotic and the near-exact quantiles, with a much more pronounced progression for the near-exact quantiles. However, for any given sample size, the near-exact quantiles, opposite to the asymptotic quantiles, as p increases, they match more correct digits.

The asymptotic and simulated quantiles exchange positions in the precision scale when the sample size increases, with the simulated quantiles showing a better precision than the asymptotic for small sample sizes and vice-versa for larger sample sizes. Anyway, the near-exact quantiles, even for only $s = 2$ exact moments matched, always show a much better performance than both the simulated and the asymptotic quantiles, what actually is in complete accordance with the values of the measure Δ in Table 1.

For all values of p, the asymptotic quantiles do not even match a single significant digit for the smaller sample sizes, that is when the sample size only exceeds p by 1.

For $p = 20$, even for quite large sample sizes, it seems that it starts becoming hard for the asymptotic quantiles to compete with the simulated ones, but there is absolutely no problem with the near-exact quantiles, which, as expected, even show an asymptotic behavior for increasing p.

Table 4 – 0.05 and 0.01 asymptotic, simulated and near-exact quantiles of Λ_1 for $p = 5$.

$p=5$ $n=5$		0.05	0.01
Asymptotic	$10^{-5}\times$	0.1	0.02
Simulated	$10^{-5}\times$	1.336	0.052
Near-exact 2	$10^{-5}\times$	1.33423	0.050631
Near-exact 4	$10^{-5}\times$	1.334235223	0.05063108169
Near-exact 6	$10^{-5}\times$	1.33423522396	0.050631081622
Near-exact 10	$10^{-5}\times$	1.33423522393393	0.0506310816254121
$p=5$ $n=10$		0.05	0.01
Asymptotic	$10^{-2}\times$	3.48	1.7
Simulated	$10^{-2}\times$	3.408	1.637
Near-exact 2	$10^{-2}\times$	3.407284	1.63722
Near-exact 4	$10^{-2}\times$	3.407285595	1.637220093
Near-exact 6	$10^{-2}\times$	3.40728559796	1.637220099132
Near-exact 10	$10^{-2}\times$	3.40728559796347	1.63722009913482
$p=5$ $n=50$		0.05	0.01
Asymptotic	$10^{-1}\times$	5.65526	5.
Simulated	$10^{-1}\times$	5.655	5.0
Near-exact 2	$10^{-1}\times$	5.655206550	5.013889
Near-exact 4	$10^{-1}\times$	5.6552065509427	5.0138894761
Near-exact 6	$10^{-1}\times$	5.6552065509423699	5.01388947613679
Near-exact 10	$10^{-1}\times$	5.655206550942369273597	5.013889476136786005098
$p=5$ $n=100$		0.05	0.01
Asymptotic	$10^{-1}\times$	7.555575	7.121559
Simulated	$10^{-1}\times$	7.556	7.124
Near-exact 2	$10^{-1}\times$	7.5555711581	7.121553347
Near-exact 4	$10^{-1}\times$	7.55557115821431	7.1215533496785
Near-exact 6	$10^{-1}\times$	7.55557115821430118	7.12155334967862291
Near-exact 10	$10^{-1}\times$	7.5555711582143011797944934	7.121553349678622923375119

Table 5 – 0.05 and 0.01 asymptotic, simulated and near-exact quantiles of Λ_1 for $p = 10$.

$p=10$ $n=10$		0.05	0.01
Asymptotic	$10^{-6}\times$	3.3	0.8
Simulated	$10^{-8}\times$	4.53	0.167
Near-exact 2	$10^{-8}\times$	4.5097776	0.16639781
Near-exact 4	$10^{-8}\times$	4.50977794422	0.166397821368
Near-exact 6	$10^{-8}\times$	4.509777944229099	0.16639782136722444
Near-exact 10	$10^{-8}\times$	4.50977794422909771042	0.166397821367224422478
$p=10$ $n=50$		0.05	0.01
Asymptotic	$10^{-1}\times$	1.879	1.541
Simulated	$10^{-1}\times$	1.88	1.542
Near-exact 2	$10^{-1}\times$	1.878800272	1.54041369
Near-exact 4	$10^{-1}\times$	1.87880027252870	1.5404136916555
Near-exact 6	$10^{-1}\times$	1.878800272528748910	1.54041369165560216
Near-exact 10	$10^{-1}\times$	1.8788002725287489168299893	1.540413691655602168398140
$p=10$ $n=100$		0.05	0.01
Asymptotic	$10^{-1}\times$	4.4667	4.05969
Simulated	$10^{-1}\times$	4.467	4.06
Near-exact 2	$10^{-1}\times$	4.4666755382	4.0596144772
Near-exact 4	$10^{-1}\times$	4.466675538319666	4.05961447773410
Near-exact 6	$10^{-1}\times$	4.46667553831966816881	4.0596144777341041323
Near-exact 10	$10^{-1}\times$	4.466675538319668168873947279	4.05961447773410413247304018
$p=10$ $n=200$		0.05	0.01
Asymptotic	$10^{-1}\times$	6.729257	6.42063
Simulated	$10^{-1}\times$	6.7295	6.4205
Near-exact 2	$10^{-1}\times$	6.72925224163	6.4206280670
Near-exact 4	$10^{-1}\times$	6.7292522416411725	6.4206280670218228
Near-exact 6	$10^{-1}\times$	6.729252241641172518157	6.42062806702182288915
Near-exact 10	$10^{-1}\times$	6.7292522416411725181571225656638	6.420628067021822889150043484803

Table 6 – 0.05 and 0.01 asymptotic, simulated and near-exact quantiles of Λ_1 for $p = 15$.

$p=15$ $n=15$		0.05	0.01
Asymptotic	$10^{-8}\times$	7.6	2.3
Simulated	$10^{-10}\times$	1.917	0.0691
Near-exact 2	$10^{-10}\times$	1.91996585	0.0695434543
Near-exact 4	$10^{-10}\times$	1.9199658564262	0.06954345480219
Near-exact 6	$10^{-10}\times$	1.91996585642633442	0.06954345480218604
Near-exact 10	$10^{-10}\times$	1.9199658564263344086778 8	0.0695434548021860398307916 9
$p=15$ $n=50$		0.05	0.01
Asymptotic	$10^{-2}\times$	3.2	2.4
Simulated	$10^{-2}\times$	3.19	2.4
Near-exact 2	$10^{-2}\times$	3.189503691	2.399300804
Near-exact 4	$10^{-2}\times$	3.18950369130166	2.39930080592976
Near-exact 6	$10^{-2}\times$	3.189503691301678567	2.399300805929779421 4
Near-exact 10	$10^{-2}\times$	3.18950369130167856863157 05	2.39930080592977942147559216
$p=15$ $n=100$		0.05	0.01
Asymptotic	$10^{-1}\times$	1.97	1.7237
Simulated	$10^{-1}\times$	1.9690	1.722
Near-exact 2	$10^{-1}\times$	1.96999393551	1.7234487162
Near-exact 4	$10^{-1}\times$	1.969993935541246	1.72344871631981
Near-exact 6	$10^{-1}\times$	1.96999393554124618129 0	1.72344871631981022565 1
Near-exact 10	$10^{-1}\times$	1.96999393554124618129466505046	1.72344871631981022565283746963
$p=15$ $n=200$		0.05	0.01
Asymptotic	$10^{-1}\times$	4.53384	4.2483
Simulated	$10^{-1}\times$	4.534	4.2486
Near-exact 2	$10^{-1}\times$	4.533819021214	4.2482843981
Near-exact 4	$10^{-1}\times$	4.53381902121811602	4.24828439810631897
Near-exact 6	$10^{-1}\times$	4.53381902121811603336791	4.24828439810631898356976
Near-exact 10	$10^{-1}\times$	4.53381902121811603336796026225176	4.24828439810631898356978458406892 2

Table 7 – 0.05 and 0.01 asymptotic, simulated and near-exact quantiles of Λ_1 for $p = 20$.

$p=20$ $n=20$		0.05	0.01
Asymptotic	$10^{-9}\times$	1.8	0.6
Simulated	$10^{-13}\times$	9.96	0.36
Near-exact 2	$10^{-13}\times$	9.99098935	0.3568991575
Near-exact 4	$10^{-13}\times$	9.99098936243010	0.356899157961628
Near-exact 6	$10^{-13}\times$	9.990989362430128835	0.35689915796162607183
Near-exact 10	$10^{-13}\times$	9.9909893624301288331974 27214	0.35689915796162607180939486 98
$p=20$ $n=50$		0.05	0.01
Asymptotic	$10^{-3}\times$	2.66	1.8
Simulated	$10^{-3}\times$	2.618	1.79
Near-exact 2	$10^{-3}\times$	2.6199149636	1.7930725171
Near-exact 4	$10^{-3}\times$	2.61991496392921	1.793072517821540
Near-exact 6	$10^{-3}\times$	2.6199149639292129111	1.7930725178215428689
Near-exact 10	$10^{-3}\times$	2.61991496392921291124741785 6	1.79307251782154286885032407 7
$p=20$ $n=100$		0.05	0.01
Asymptotic	$10^{-2}\times$	6.515	5.48
Simulated	$10^{-2}\times$	6.515	5.48
Near-exact 2	$10^{-2}\times$	6.5120357825	5.477980148
Near-exact 4	$10^{-2}\times$	6.5120357825503517	5.4779801480498865
Near-exact 6	$10^{-2}\times$	6.51203578255035199100	5.47798014804988675576 46
Near-exact 10	$10^{-2}\times$	6.51203578255035199100009928316	5.47798014804988675576401415591
$p=20$ $n=200$		0.05	0.01
Asymptotic	$10^{-1}\times$	2.6807	2.46669
Simulated	$10^{-1}\times$	2.6809	2.4661
Near-exact 2	$10^{-1}\times$	2.680689375754	2.466627224350
Near-exact 4	$10^{-1}\times$	2.680689375755469103	2.466627224355018610
Near-exact 6	$10^{-1}\times$	2.68068937575546910443470	2.46662722435501861163414 7
Near-exact 10	$10^{-1}\times$	2.68068937575546910443471066407117 3	2.46662722435501861163414582936523 7

The fact that the values in Tables 2 and 3 are all lower than the corresponding values in Table 1, that is, the values in Table 1 for the same values of n and p, as it was indeed supposed to be, confirms that the quantiles in Tables 4-7 have the required precision.

In Table 8 we have the computing times for the simulated, asymptotic and near-exact 0.05 quantiles for all the combinations of values of p and n that we considered so far in our numerical studies. These computing times were obtained for the same machine and softwares described at the end of subsection 2.2. We may see how the computation times for the asymptotic quantiles are really extremely low for all cases. Anyway, the best balance, which combines a very high precision with good computing times is without any doubt given by the near-exact quantiles, mainly for the smaller sample sizes. Even for the near-exact distributions which match 10 exact moments and which give extremely small errors, the computation times remain much lower than the computation times for the simulated quantiles.

Table 8 – Computing times in seconds for the simulation (samples of size $2{\times}10^6$), asymptotic and near-exact 0.05 quantiles.

p	n	Simul.	Box	Near-exact distributions number of exact moments equated			
				2	4	6	10
5	5	23.4	0.1	0.3	0.4	0.6	0.9
	10	31.1	0.1	0.2	0.4	0.5	0.8
	50	31.2	0.1	0.2	0.3	0.5	0.8
	100	31.3	0.1	0.2	0.3	0.4	0.9
10	10	58.7	0.1	1.5	2.4	3.7	6.8
	50	66.1	0.1	1.5	2.6	4.0	7.1
	100	66.9	0.1	1.4	2.5	3.6	6.4
	200	67.4	0.1	1.4	2.8	3.7	6.6
15	15	94.5	0.1	6.3	11.1	15.9	22.4
	50	103.3	0.1	6.8	11.5	16.2	23.4
	100	103.8	0.1	6.5	11.2	16.2	26.6
	200	104.4	0.1	6.5	11.5	16.3	24.7
20	20	125.8	0.1	15.5	25.9	34.3	51.3
	50	134.6	0.1	22.6	39.1	53.0	78.4
	100	140.4	0.1	25.0	40.7	53.9	79.8
	200	141.6	0.2	24.8	41.2	54.5	80.2

5. Final remarks

We have shown how:

- by working through the c.f. of $W_2 = -\log \Lambda_2$, where Λ_2 is the l.r.t. statistic in (15), it is possible to obtain the exact distribution of W_2 as a GIG distribution and consequently obtain simple closed forms for the exact p.d.f. and c.d.f. of Λ_2;
- by working through the c.f. of $W_1 = -\log \Lambda_1$, where Λ_1 is the l.r.t. statistic in (5), and using a decomposition of the ratio of Gamma functions derived from a result from Tricomi and Erdélyi [4], it is possible to obtain the exact distribution of W_1 as an infinite mixture of GNIG distributions.

Then, from this representation of the exact distribution of Λ_1 it was almost easy to obtain very well-fitting near-exact distributions both for W_1 and Λ_1. These near-exact distributions are, relatively to W_1, finite mixtures of GNIG distributions, which, by construction, equate a given number of exact moments. They are very easy to implement computationally and allow for an easy and quick computation of near-exact quantiles and p-values, which may indeed, given their closeness to the exact ones, be used instead of these ones.

We may see that, as expected the near-exact distributions show a much better performance than the asymptotic distribution, with increasing performance for increasing number of exact moments matched. Also, the near-exact distributions show a clear asymptotic behavior both for increasing sample sizes and increasing number of variables involved, with already extremely good performances for small sample sizes.

Appendices

Appendix A. Notation used in the expressions of the l.r.t. statistics

In this Appendix we summarize the main results in [1] and establish the notation related with the l.r.t. statistics used in this paper.

Let $\boldsymbol{\Sigma}_c$ be as in (1)-(3). Then, according to [1], since $\boldsymbol{\Sigma}_c$ is symmetric its eigenvalues are real and there exists an orthogonal matrix $\mathbf{P}$, such that,

$$\boldsymbol{\Sigma}_c = \mathbf{P}\mathbf{D}_\lambda\mathbf{P}'$$

with $\mathbf{D}_\lambda = diag\,(\lambda_1, ..., \lambda_p)$. The columns of the matrix $\mathbf{P} = [\mathbf{u}_{jk}]$ are the eigenvectors of $\boldsymbol{\Sigma}_c$, corresponding to $\lambda_1, ..., \lambda_p$, and may be given by

$$\mathbf{u}_{jk} = \frac{1}{\sqrt{p}}\left\{cos\left[\frac{2\pi}{p}(j-1)(k-1)\right] + sin\left[\frac{2\pi}{p}(j-1)(k-1)\right]\right\}, \quad j,k = 1, ..., p\,.$$

It is interesting to note that the $\mathbf{u}_{jk}$ indeed do not depend on the elements of $\boldsymbol{\Sigma}_c$, only the eigenvalues $\lambda_1, \ldots, \lambda_p$ will do (see [1] for details).

Consider a random sample of size $N = n+1$ from the distribution $\mathcal{N}_p\,(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and let $\mathbf{X}_{N\times p}$ be the sample matrix. Let $\mathbf{E}_{N1}$ be an $N \times 1$ unitary vector. Let

$$\overline{\mathbf{X}} = \left[\overline{\mathbf{X}}_1 \ldots \overline{\mathbf{X}}_p\right] = \frac{1}{N}\mathbf{X}'\mathbf{E}_{N1}$$

be the vector of sample means, and

$$\mathbf{S} = \left(\mathbf{X} - \mathbf{E}_{N1}\overline{\mathbf{X}}'\right)'\left(\mathbf{X} - \mathbf{E}_{N1}\overline{\mathbf{X}}'\right)\,.$$

Let us take $\mathbf{y} = \sqrt{N}\,\overline{\mathbf{X}}\mathbf{P}$ and $\mathbf{V} = \mathbf{P}'\mathbf{S}\mathbf{P}$. Then, $\mathbf{y}$ and $\mathbf{V} = [v_{ij}]$ are independently distributed with

$$\mathbf{y} \sim \mathcal{N}_p\left(\sqrt{N}\boldsymbol{\mu}\mathbf{P}, \overline{\boldsymbol{\Sigma}}\right) \qquad\qquad \mathbf{V} \sim \mathcal{W}_p\left(N-1, \overline{\boldsymbol{\Sigma}}\right)$$

where $\overline{\boldsymbol{\Sigma}} = \mathbf{P}'\boldsymbol{\Sigma}\mathbf{P}$. If $\boldsymbol{\Sigma}$ is circular then

$$\overline{\boldsymbol{\Sigma}} = \mathbf{D}_\lambda = diag\,(\lambda_1, ..., \lambda_p)\,, \quad \text{with} \;\; \lambda_j = \lambda_{p-j+2}\,, \quad \text{for} \;\; j = 2, ..., p\,.$$

We then define, for even p,

$$v_j = \begin{cases} v_{jj}, & j = 1 \text{ or } j = m+1 \\ v_{jj} + v_{p-j+2,p-j+2}, & j = 2, ..., m, \end{cases} \tag{A1}$$

while for odd p,

$$v_j = \begin{cases} v_{jj}, & j = 1 \\ v_{jj} + v_{p-j+2,p-j+2}, & j = 2, ..., m+1, \end{cases} \tag{A2}$$

with $v_{p-j+2} = v_j$ for $(j = 2, \dots, p)$, and, for even p,

$$w_j = \begin{cases} y_j^2, & j = 1 \text{ or } j = m+1 \\ y_j^2 + y_{p-j+2}^2, & j = 2, ..., m, \end{cases} \tag{A3}$$

while for odd p,

$$w_j = \begin{cases} y_1^2, & j = 1 \\ y_j^2 + y_{p-j+2}^2, & j = 2, ..., m+1. \end{cases} \tag{A4}$$

Appendix B. The Gamma, GIG (Generalized Integer Gamma) and GNIG (Generalized Near-Integer Gamma) Distributions

If a random variable X has density given by

$$f_X(x) = \frac{\lambda^r}{\Gamma(r)}\, e^{-\lambda x} x^{r-1}, \quad x > 0 \tag{B1}$$

then it is defined to have a Gamma distribution with shape parameter $r > 0$ and rate parameter $\lambda > 0$ and we will denote this fact by

$$X \sim \Gamma(r, \lambda).$$

Let $X_1, ..., X_p$ be p independent random variables with

$$X_j \sim \Gamma\left(r_j, \lambda_j\right), \qquad r_j \in \quad , \qquad (j = 1, ..., p)$$

with $\lambda_i \neq \lambda_j$, for all $i, j \in \{1, ..., p\}$. Then the random variable

$$Y = \sum_{j=1}^{p} X_j$$

has a GIG (Generalized Integer Gamma) distribution of depth p (see [3]), with shape parameters r_j and rate parameters λ_j $(j = 1, ...p)$. We will denote this fact by

$$Y \sim GIG\left(r_1, ..., r_p; \lambda_1, ..., \lambda_p; p\right).$$

The p.d.f. and c.d.f. of Y are respectively given, for $y > 0$, by

$$f_Y^{GIG}\left(y | r_1, ..., r_p; \lambda_1, ..., \lambda_p; p\right) = K \sum_{j=1}^{p} \left\{ \sum_{k=1}^{r_j} c_{jk} y^{k-1} \right\} e^{-\lambda_j y},$$

and

$$F_Y^{GIG}\left(y | r_1, ..., r_p; \lambda_1, ..., \lambda_p; p\right) = 1 - K \sum_{j=1}^{p} \left\{ \sum_{k=1}^{r_j} c_{jk} (k-1)! \sum_{i=0}^{k-1} \frac{y^i}{i!\, \lambda_j^{k-i}} \right\} e^{-\lambda_j y},$$

where K by given (5) in [3] and the c_{jk} are given by (11)–(13) in the same reference.
Let then Y and X be independent random variables, such that,

$$Y \sim GIG\left(r_1, ..., r_p; \lambda_1, ..., \lambda_p, ; p\right) \quad \textit{and} \quad X \sim \Gamma(r, \lambda)$$

with $r \in \ ^{+}\backslash$ and $\lambda \neq \lambda_j$, for all $j \in \{1, ..., p\}$. Then, the random variable

$$W = Y + X$$

has a GNIG (Generalized Near-Integer Gamma) distribution of depth $p+1$ (see [2]),

$$W \sim GNIG\left(r, r_1, ..., r_p; \lambda, \lambda_1, ..., \lambda_p; p+1\right).$$

The p.d.f. and c.d.f. of W are, for $w > 0$,

$$f_W^{GNIG}(w|r_1, \dots, r_p; r; \lambda_1, \dots, \lambda_p; \lambda; p+1) = \\ K\lambda^r \sum_{j=1}^{p} e^{-\lambda_j w} \sum_{k=1}^{r_j} \left\{ c_{jk} \frac{\Gamma(k)}{\Gamma(k+r)}\, w^{k+r-1}\, {}_1F_1(r, k+r, -(\lambda-\lambda_j)w \right\}$$

and

$$F_W^{GNIG}(w|r_1, \dots, r_p; r; \lambda_1, \dots, \lambda_p; \lambda; p+1) = \frac{\lambda^r\, w^r}{\Gamma(r+1)}\, {}_1F_1(r, r+1, -\lambda w) \\ -K\lambda^r \sum_{j=1}^{p} e^{-\lambda_j w} \sum_{k=1}^{r_j} c_{jk}^* \sum_{i=0}^{k-1} \frac{w^{r+i}\lambda_j^i}{\Gamma(r+1+i)}\, {}_1F_1\left(r, r+1+i, -(\lambda-\lambda_j)w\right)\,,$$

where K is given by (5) in [3] and

$$c_{jk}^* = \frac{c_{jk}}{\lambda_j^k}\, \Gamma(k)$$

with c_{jk} given by (11)–(13) in [3].

If $r \in$, the GNIG distribution of depth $p+1$ reduces to the GIG distribution of depth $p+1$. We can thus consider the distribution GNIG as a generalization of the GIG distribution.

B.1. *Expressing a Logbeta as a mixture of Gammas*

From the two first expressions in section 5 of [4] and also expressions (11) and (14) in the same paper, we may write

$$\frac{\Gamma(a-it)}{\Gamma(a+b-it)} = \sum_{j=0}^{\infty} p_j(b)(a-it)^{-b-j} \tag{B2}$$

where $p_j(b)$ is a polynomial of degree j in b with

$$p_j(b) = \frac{1}{j} \sum_{m=0}^{j-1} \left(\frac{\Gamma(1-b-m)}{\Gamma(-b-j)(j-m+1)!} + (-1)^{j+m} b^{j-m+1} \right) p_m(b) \tag{B3}$$

where

$$p_0(b) = 1. \tag{B4}$$

Then, since the c.f. of $Y = -\log X$, where $X \sim Beta(a, b)$, is given by

$$\Phi_Y(t) = \frac{\Gamma(a+b)}{\Gamma(a)} \frac{\Gamma(a-it)}{(a+b-it)}\,, \tag{B5}$$

using (B2), we may write

$$\Phi_Y(t) = \sum_{j=0}^{\infty} \underbrace{\frac{\Gamma(a+b)}{\Gamma(a)} \frac{p_j(b)}{a^{b+j}}}_{p_j^*} \, a^{b+j} (a-it)^{-(b+j)}\,, \tag{B6}$$

which is the c.f. of an infinite mixture of $\Gamma(b+j, a)$ distributions, with weights p_j^*, with p_j given by (B3) and (B4).

Appendix C. Exact quantiles for Λ_2 in (15)

Table C.1 – 0.05 exact quantiles for Λ_2 for $p = 3, 4, 5, 6$ and samples of size $n+1 = n^*+p$ and $n+1 = N$.

	p = 3	4	5	6
n^*				
1	$2.873847984290^{(-4)}$	$7.302419577579^{(-5)}$	$1.704898981532^{(-5)}$	$5.310939244845^{(-6)}$
2	$9.527997686826^{(-3)}$	$3.079560811537^{(-3)}$	$8.767838828202^{(-4)}$	$3.126339693497^{(-4)}$
3	$3.581759788911^{(-2)}$	$1.384791167079^{(-2)}$	$4.635541484684^{(-3)}$	$1.855580841869^{(-3)}$
4	$7.362451240167^{(-2)}$	$3.250347551847^{(-2)}$	$1.236608135859^{(-2)}$	$5.442391973769^{(-3)}$
5	$1.164593678068^{(-1)}$	$5.687956019752^{(-2)}$	$2.397128595067^{(-2)}$	$1.140542320428^{(-2)}$
6	$1.602485851264^{(-1)}$	$8.471340429834^{(-2)}$	$3.879965395634^{(-2)}$	$1.969807855114^{(-2)}$
7	$2.028226438591^{(-1)}$	$1.142568856233^{(-1)}$	$5.606046373144^{(-2)}$	$3.006073945520^{(-2)}$
8	$2.431463268932^{(-1)}$	$1.443015602244^{(-1)}$	$7.501670387195^{(-2)}$	$4.214614307815^{(-2)}$
9	$2.808080002982^{(-1)}$	$1.740606707355^{(-1)}$	$9.505221485266^{(-2)}$	$5.559437015579^{(-2)}$
10	$3.157255020470^{(-1)}$	$2.030458892282^{(-1)}$	$1.156806231829^{(-1)}$	$7.007130729848^{(-2)}$
N				
20	$4.958948035216^{(-1)}$	$3.517971884030^{(-1)}$	$2.176901262207^{(-1)}$	$1.330994309961^{(-1)}$
25	$5.779024950788^{(-1)}$	$4.438941476742^{(-1)}$	$3.071437423092^{(-1)}$	$2.119154973043^{(-1)}$
30	$6.375054947603^{(-1)}$	$5.145516432538^{(-1)}$	$3.815915521697^{(-1)}$	$2.832094073562^{(-1)}$
35	$6.825877572039^{(-1)}$	$5.699149470293^{(-1)}$	$4.431131798471^{(-1)}$	$3.453540109436^{(-1)}$
40	$7.178069667847^{(-1)}$	$6.142542514095^{(-1)}$	$4.942699250727^{(-1)}$	$3.989824205411^{(-1)}$
50	$7.691859983290^{(-1)}$	$6.805688068408^{(-1)}$	$5.737273574584^{(-1)}$	$4.854168081311^{(-1)}$

Table C.2 – 0.05 exact quantiles for Λ_2 for $p = 7, 8, 9, 10$ and samples of size $n+1 = n^*+p$ and $n+1 = N$.

	p = 7	8	9	10
n^*				
1	$1.475636237066^{(-6)}$	$4.899436215304^{(-7)}$	$1.472087368039^{(-7)}$	$5.035523819189^{(-8)}$
2	$9.866579284725^{(-5)}$	$3.611361971684^{(-5)}$	$1.193402368187^{(-5)}$	$4.410573164100^{(-6)}$
3	$6.558115113440^{(-4)}$	$2.623369863805^{(-4)}$	$9.471371135654^{(-5)}$	$3.765737319204^{(-5)}$
4	$2.116490969784^{(-3)}$	$9.140206142033^{(-4)}$	$3.566467087933^{(-4)}$	$1.513195279785^{(-4)}$
5	$4.807633447564^{(-3)}$	$2.217106852995^{(-3)}$	$9.256133058325^{(-4)}$	$4.158565390085^{(-4)}$
6	$8.891065053167^{(-3)}$	$4.338872568176^{(-3)}$	$1.921536135091^{(-3)}$	$9.080001408914^{(-4)}$
7	$1.438784077253^{(-2)}$	$7.374665801665^{(-3)}$	$3.439629798034^{(-3)}$	$1.699690651153^{(-3)}$
8	$2.122219236876^{(-2)}$	$1.135504856668^{(-2)}$	$5.544121775464^{(-3)}$	$2.850922715506^{(-3)}$
9	$2.926182105524^{(-2)}$	$1.626062103356^{(-2)}$	$8.268903222091^{(-3)}$	$4.406402157399^{(-3)}$
10	$3.834791975939^{(-2)}$	$2.203753781050^{(-2)}$	$1.162175615039^{(-2)}$	$6.395017624079^{(-3)}$
N				
25	$1.311174631016^{(-1)}$	$7.991513087230^{(-2)}$	$4.339763565552^{(-2)}$	$2.294641180236^{(-2)}$
30	$1.926863368273^{(-1)}$	$1.301029968927^{(-1)}$	$8.026650042403^{(-2)}$	$4.879545105565^{(-2)}$
35	$2.503469518940^{(-1)}$	$1.807623468136^{(-1)}$	$1.211737053908^{(-1)}$	$8.050886894665^{(-2)}$
40	$3.026527064946^{(-1)}$	$2.291399412701^{(-1)}$	$1.628531601524^{(-1)}$	$1.150899076343^{(-1)}$
50	$3.912716594270^{(-1)}$	$3.153883944175^{(-1)}$	$2.421104789760^{(-1)}$	$1.854080894471^{(-1)}$
100	$6.359084503274^{(-1)}$	$5.745064869794^{(-1)}$	$5.072638615384^{(-1)}$	$4.481097642943^{(-1)}$

Table C.3 – 0.05 exact quantiles for Λ_2 for $p = 12, 14, 16, 18$ and samples of size $n{+}1{=}n^*{+}p$ and $n{+}1{=}N$.

	p			
	12	14	16	18
n^*				
1	$5.502348448406^{(-9)}$	$6.253393163854^{(-10)}$	$7.303478498494^{(-11)}$	$8.702312606453^{(-12)}$
2	$5.549385790651^{(-7)}$	$7.109110867373^{(-8)}$	$9.215567932974^{(-9)}$	$1.204498140261^{(-9)}$
3	$5.426007898658^{(-6)}$	$7.815525474343^{(-7)}$	$1.123554326117^{(-7)}$	$1.611154384193^{(-8)}$
4	$2.464600149512^{(-5)}$	$3.953874654499^{(-6)}$	$6.258816266054^{(-7)}$	$9.792569748530^{(-8)}$
5	$7.557175756141^{(-5)}$	$1.336941408545^{(-5)}$	$2.312102438591^{(-6)}$	$3.922114871419^{(-7)}$
6	$1.819690948056^{(-4)}$	$3.517502178561^{(-5)}$	$6.596694309647^{(-6)}$	$1.205825536995^{(-6)}$
7	$3.718446998796^{(-4)}$	$7.789698297350^{(-5)}$	$1.573527129967^{(-5)}$	$3.081876952726^{(-6)}$
8	$6.748976780970^{(-4)}$	$1.521165262442^{(-4)}$	$3.289716585842^{(-5)}$	$6.868217714845^{(-6)}$
9	$1.120195438454^{(-3)}$	$2.699261495967^{(-4)}$	$6.215900685154^{(-5)}$	$1.376927422901^{(-5)}$
10	$1.734372581794^{(-3)}$	$4.442915319867^{(-4)}$	$1.084189243447^{(-4)}$	$2.537511049318^{(-5)}$
N				
30	$1.458385339039^{(-2)}$	$3.372678492778^{(-3)}$	$5.812336052188^{(-4)}$	$7.099290514329^{(-5)}$
35	$3.004891894319^{(-2)}$	$9.235149697937^{(-3)}$	$2.288608266888^{(-3)}$	$4.456399812626^{(-4)}$
40	$5.002208314646^{(-2)}$	$1.859556512352^{(-2)}$	$5.833969307742^{(-3)}$	$1.520516878100^{(-3)}$
45	$7.309303454959^{(-2)}$	$3.115326619152^{(-2)}$	$1.154325685519^{(-2)}$	$3.679201004160^{(-3)}$
50	$9.802812265422^{(-2)}$	$4.632268986075^{(-2)}$	$1.943087209717^{(-2)}$	$7.180296457357^{(-3)}$
100	$3.340044962156^{(-1)}$	$2.376152967380^{(-1)}$	$1.611513115608^{(-2)}$	$1.040647188578^{(-1)}$

Table C.4 – 0.05 exact quantiles for Λ_2 for $p = 20, 25, 30, 50$ and samples of size $n + 1 = n^* + p$ and $n + 1 = N$.

	p			
	20	25	30	50
n^*				
1	$1.052884280853^{(-12)}$	$5.512772564595^{(-15)}$	$3.147950236676^{(-17)}$	$3.922126036501^{(-26)}$
2	$1.583775535321^{(-10)}$	$9.912072745346^{(-13)}$	$6.551694538971^{(-15)}$	$1.241681170431^{(-23)}$
3	$2.304221327768^{(-9)}$	$1.729905292316^{(-11)}$	$1.329423144993^{(-13)}$	$3.901547276512^{(-22)}$
4	$1.516652720297^{(-8)}$	$1.355475776810^{(-10)}$	$1.205397668992^{(-12)}$	$5.436447268532^{(-21)}$
5	$6.544040621866^{(-8)}$	$6.894955410377^{(-10)}$	$7.048407303498^{(-12)}$	$4.820342534479^{(-20)}$
6	$2.156311329162^{(-7)}$	$2.652114165079^{(-9)}$	$3.094950692108^{(-11)}$	$3.161964615187^{(-19)}$
7	$5.878268392178^{(-7)}$	$8.360501214938^{(-9)}$	$1.106226311900^{(-10)}$	$1.662974912924^{(-18)}$
8	$1.391102346721^{(-6)}$	$2.267872016742^{(-8)}$	$3.380407875168^{(-10)}$	$7.367263619085^{(-18)}$
9	$2.949498801798^{(-6)}$	$5.466905092668^{(-8)}$	$9.123983888815^{(-10)}$	$2.841869508075^{(-17)}$
10	$5.727505700064^{(-6)}$	$1.197911495577^{(-7)}$	$2.225780558649^{(-9)}$	$9.773284741921^{(-17)}$
N				
50	$2.317475273464^{(-3)}$	$7.012512095672^{(-5)}$	$7.112006098236^{(-7)}$	—
60	$7.579560432640^{(-3)}$	$4.987398360175^{(-4)}$	$1.545989220256^{(-5)}$	$9.773284741921^{(-17)}$
70	$1.677744976099^{(-2)}$	$1.802115336452^{(-3)}$	$1.090689208238^{(-4)}$	$5.212704133568^{(-13)}$
80	$2.969227689185^{(-2)}$	$4.475395356308^{(-3)}$	$4.244475302843^{(-4)}$	$8.492794158579^{(-11)}$
90	$4.565999315106^{(-2)}$	$8.825312749519^{(-3)}$	$1.156732761041^{(-3)}$	$2.714484142577^{(-9)}$
100	$6.390539981177^{(-2)}$	$1.494439385149^{(-2)}$	$2.501550255200^{(-3)}$	$3.420944640557^{(-8)}$

Table C.5 – 0.01 exact quantiles for Λ_2 for $p = 3, 4, 5, 6$ and samples of size $n{+}1{=}n^*{+}p$ and $n{+}1{=}N$.

	p			
	3	4	5	6
$n^* = 1$	$1.118572431319^{(-5)}$	$2.805677881029^{(-6)}$	$6.482035993955^{(-7)}$	$2.004539366120^{(-7)}$
2	$1.763885394084^{(-3)}$	$5.471409336487^{(-4)}$	$1.503804565295^{(-4)}$	$5.230657654407^{(-5)}$
3	$1.115977654543^{(-2)}$	$4.095300497286^{(-3)}$	$1.308376341352^{(-3)}$	$5.065961412232^{(-4)}$
4	$2.995320779750^{(-2)}$	$1.251988108179^{(-2)}$	$4.528187660677^{(-3)}$	$1.921382704775^{(-3)}$
5	$5.584882682469^{(-2)}$	$2.584709090146^{(-2)}$	$1.034941397292^{(-2)}$	$4.743518727794^{(-3)}$
6	$8.598411080417^{(-2)}$	$4.316311392292^{(-2)}$	$1.880020036661^{(-2)}$	$9.197282049541^{(-3)}$
7	$1.181196992673^{(-1)}$	$6.334736183255^{(-2)}$	$2.960568258191^{(-2)}$	$1.531084341103^{(-2)}$
8	$1.507436076636^{(-1)}$	$8.539080267533^{(-2)}$	$4.236301662083^{(-2)}$	$2.298062599980^{(-2)}$
9	$1.829077751471^{(-1)}$	$1.084872990563^{(-1)}$	$5.664708675824^{(-2)}$	$3.203034565261^{(-2)}$
10	$2.140515789328^{(-1)}$	$1.320312717283^{(-1)}$	$7.206375559008^{(-2)}$	$4.225347879875^{(-2)}$
$N = 20$	$3.918073550003^{(-1)}$	$2.656958191068^{(-1)}$	$1.561019794662^{(-1)}$	$9.089963140953^{(-2)}$
25	$4.807617068603^{(-1)}$	$3.569732034087^{(-1)}$	$2.375624682910^{(-1)}$	$1.581848665210^{(-1)}$
30	$5.481437514406^{(-1)}$	$4.305771564955^{(-1)}$	$3.094919642674^{(-1)}$	$2.233620880967^{(-1)}$
35	$6.005343214231^{(-1)}$	$4.901827408608^{(-1)}$	$3.713000158942^{(-1)}$	$2.827837909288^{(-1)}$
40	$6.422772201966^{(-1)}$	$5.390528884362^{(-1)}$	$4.241390588334^{(-1)}$	$3.357143708078^{(-1)}$
50	$7.044027298143^{(-1)}$	$6.139015531083^{(-1)}$	$5.085472392043^{(-1)}$	$4.238072240871^{(-1)}$

Table C.6 – 0.01 exact quantiles for Λ_2 for $p = 7, 8, 9, 10$ and samples of size $n+1=n^*+p$ and $n+1=N$.

	p			
n^*	7	8	9	10
1	$5.530910674520^{(-8)}$	$1.826441476486^{(-8)}$	$5.458325025425^{(-9)}$	$1.858852075637^{(-9)}$
2	$1.611814854278^{(-5)}$	$5.790982256733^{(-6)}$	$1.878892622394^{(-6)}$	$6.840275361857^{(-7)}$
3	$1.733394808441^{(-4)}$	$6.760759671977^{(-5)}$	$2.380482931088^{(-5)}$	$9.271795397693^{(-6)}$
4	$7.207873115152^{(-4)}$	$3.025951611369^{(-4)}$	$1.147885085194^{(-4)}$	$4.758428423887^{(-5)}$
5	$1.926384117500^{(-3)}$	$8.625853922204^{(-4)}$	$3.496307496683^{(-4)}$	$1.532857842886^{(-4)}$
6	$3.999300846648^{(-3)}$	$1.894563065234^{(-3)}$	$8.142634571919^{(-4)}$	$3.753061461931^{(-4)}$
7	$7.063669728608^{(-3)}$	$3.515584800171^{(-3)}$	$1.591471063043^{(-3)}$	$7.670717680380^{(-4)}$
8	$1.116373336107^{(-2)}$	$5.803204168629^{(-3)}$	$2.751193320662^{(-3)}$	$1.380258657369^{(-3)}$
9	$1.628156051740^{(-2)}$	$8.796203656658^{(-3)}$	$4.345776979140^{(-3)}$	$2.260290667576^{(-3)}$
10	$2.235659824881^{(-2)}$	$1.250047403590^{(-2)}$	$6.408986871068^{(-3)}$	$3.443758518251^{(-3)}$
N				
25	$9.402893632089^{(-2)}$	$5.513629472140^{(-2)}$	$2.868398827773^{(-2)}$	$1.453089591855^{(-2)}$
30	$1.472560217489^{(-1)}$	$9.650917968604^{(-2)}$	$5.761281737432^{(-2)}$	$3.391682277286^{(-2)}$
35	$1.997322557624^{(-1)}$	$1.407595341961^{(-1)}$	$9.186628940723^{(-2)}$	$5.948544793041^{(-2)}$
40	$2.490874433898^{(-1)}$	$1.847647587319^{(-1)}$	$1.283847459648^{(-1)}$	$8.880082330269^{(-2)}$
50	$3.358275766237^{(-1)}$	$2.665067054518^{(-1)}$	$2.010957954304^{(-1)}$	$1.515264290403^{(-1)}$
100	$5.907581659885^{(-1)}$	$5.299188970499^{(-1)}$	$4.642140355661^{(-1)}$	$4.071131558695^{(-1)}$

Table C.7 – 0.01 exact quantiles for Λ_2 for $p = 12, 14, 16, 18$ and samples of size $n+1=n^*+p$ and $n+1=N$.

	p			
n^*	12	14	16	18
1	$2.014526148600^{(-10)}$	$2.273185179961^{(-11)}$	$2.638090536859^{(-12)}$	$3.125388182285^{(-13)}$
2	$8.371267687365^{(-8)}$	$1.047245285406^{(-8)}$	$1.329587013559^{(-9)}$	$1.705883877078^{(-10)}$
3	$1.286268401421^{(-6)}$	$1.793559814976^{(-7)}$	$2.506271905129^{(-8)}$	$3.504482464330^{(-9)}$
4	$7.424242097953^{(-6)}$	$1.147879619932^{(-6)}$	$1.759215158077^{(-7)}$	$2.674432713158^{(-8)}$
5	$2.661782597595^{(-5)}$	$4.527700781574^{(-6)}$	$7.564547621170^{(-7)}$	$1.244319427758^{(-7)}$
6	$7.179283033455^{(-5)}$	$1.332805820388^{(-5)}$	$2.411923138032^{(-6)}$	$4.270342378763^{(-7)}$
7	$1.601389399319^{(-4)}$	$3.220385041398^{(-5)}$	$6.273720499760^{(-6)}$	$1.189446186963^{(-6)}$
8	$3.118775365935^{(-4)}$	$6.747899364156^{(-5)}$	$1.407159005905^{(-5)}$	$2.843122189796^{(-6)}$
9	$5.487578771618^{(-4)}$	$1.269683218768^{(-4)}$	$2.819601801425^{(-5)}$	$6.044374765620^{(-6)}$
10	$8.926037390899^{(-4)}$	$2.196572167153^{(-4)}$	$5.170495250990^{(-5)}$	$1.171255352899^{(-5)}$
N				
30	$9.451293396200^{(-3)}$	$2.024814189311^{(-3)}$	$3.205713506059^{(-4)}$	$3.557008420314^{(-5)}$
35	$2.100337953877^{(-2)}$	$6.083928229321^{(-3)}$	$1.414532567147^{(-3)}$	$2.569654451086^{(-4)}$
40	$3.685896285844^{(-2)}$	$1.305411948800^{(-2)}$	$3.890612842524^{(-3)}$	$9.600337559131^{(-4)}$
45	$5.600350392522^{(-2)}$	$2.290938442296^{(-2)}$	$8.131301842441^{(-3)}$	$2.476992124150^{(-3)}$
50	$7.740427311800^{(-2)}$	$3.529629178204^{(-2)}$	$1.426702229805^{(-2)}$	$5.072109563375^{(-3)}$
100	$2.988426468027^{(-1)}$	$2.093364795403^{(-1)}$	$1.397628020452^{(-1)}$	$8.882710220553^{(-2)}$

Table C.8 – 0.01 exact quantiles for Λ_2 for $p = 20, 25, 30, 50$ and samples of size $n+1 = n^* + p$ and $n+1 = N$.

	p			
n^*	20	25	30	50
1	$3.761617054839^{(-14)}$	$1.947033807610^{(-16)}$	$1.101094071414^{(-18)}$	$1.332568013181^{(-27)}$
2	$2.205843440750^{(-11)}$	$1.331794287778^{(-13)}$	$8.546999716881^{(-16)}$	$1.488840386126^{(-24)}$
3	$4.899657947012^{(-10)}$	$3.504402445946^{(-12)}$	$2.588815690934^{(-14)}$	$6.797388745596^{(-23)}$
4	$4.036279287318^{(-9)}$	$3.413136357897^{(-11)}$	$2.901507065023^{(-13)}$	$1.153089327083^{(-21)}$
5	$2.019341812920^{(-8)}$	$2.004842726106^{(-10)}$	$1.952465142594^{(-12)}$	$1.165179758352^{(-20)}$
6	$7.419342179778^{(-8)}$	$8.576859000264^{(-10)}$	$9.514114866451^{(-12)}$	$8.425479573474^{(-20)}$
7	$2.202840444747^{(-7)}$	$2.940147194201^{(-9)}$	$3.692524876268^{(-11)}$	$4.788822065381^{(-19)}$
8	$5.589562949299^{(-7)}$	$8.543661670094^{(-9)}$	$1.207601142971^{(-10)}$	$2.262762085176^{(-18)}$
9	$1.256641041404^{(-6)}$	$2.182788993933^{(-8)}$	$3.452449311158^{(-10)}$	$9.222615422506^{(-18)}$
10	$2.565972688221^{(-6)}$	$5.028752276670^{(-8)}$	$8.852165289110^{(-10)}$	$3.327777573869^{(-17)}$
N				
50	$1.572064503660^{(-3)}$	$4.255850815096^{(-5)}$	$3.789663364546^{(-7)}$	—
60	$5.552957002151^{(-3)}$	$3.362560618774^{(-4)}$	$9.505519162270^{(-6)}$	$3.327777573869^{(-17)}$
70	$1.293655277893^{(-2)}$	$1.300158159298^{(-3)}$	$7.323805433311^{(-5)}$	$2.378724936751^{(-13)}$
80	$2.374741262545^{(-2)}$	$3.386070728387^{(-3)}$	$3.027536146396^{(-4)}$	$4.517709262176^{(-11)}$
90	$3.753569473392^{(-2)}$	$6.917233477763^{(-3)}$	$8.623687977722^{(-4)}$	$1.594788089846^{(-9)}$
100	$5.367258246800^{(-2)}$	$1.203758455732^{(-2)}$	$1.929128318360^{(-3)}$	$2.157438985074^{(-8)}$

Appendix D. Modules for the computation of the exact p.d.f. and c.d.f. of Λ_2 in (15)

In this Appendix are displayed Mathematica modules for the computational implementation of the exact p.d.f. and c.d.f. of Λ_2 in (15), according to (16) and (17). The main modules are displayed in Figure D.1. These modules call the modules `GIGpdf` and `GIGcdf` in Figure D.2, which themselves both call the module `Makec` also in Figure D.2. These modules in Figure D.2 implement the p.d.f. and c.d.f. of a GIG distribution, according to the expressions in Appendix B and in [3].

```
PDFL[n_, p_, x_] := Module[{m,r,mm},
    m = Floor[p/2];
    r = If[Mod[p, 2] == 0, Table[m + Floor[-Abs[j]/2], {j, -1, p - 2}],
        Table[m - Floor[Abs[j]/2], {j, -1, p - 2}]];
    mm = Table[(n-1-j)/2, {j, 0, p - 1}];
    GIGpdf[r, mm, -Log[x]]*1/x
      ]

CDFL[n_, p_, x_] := Module[{m,r,mm},
    m = Floor[p/2];
    r = If[Mod[p, 2] == 0, Table[m + Floor[-Abs[j]/2], {j, -1, p - 2}],
        Table[m - Floor[Abs[j]/2], {j, -1, p - 2}]];
    mm = Table[(n-1-j)/2, {j, 0, p - 1}];
    1 - GIGcdf[r, mm, -Log[x]]
      ]
```

Figure D.1 – Mathematica modules for the computation of the exact p.d.f. and c.d.f. of Λ_2 in (15).

```
GIGpdf[r_, li_, z_] :=
  Module[{p,l,c},
    If[Count[r,_Integer]==Length[r]&&And@@Positive[r]&&And@@Positive[li]&&
                                                    Length[r]==Length[li],
      p = Length[r];
      l = Rationalize[li,0];
      c = Makec[r,l,p];
      P[w_]:=Table[Sum[c[[j]][[k]]*w^(k-1),{k,1,r[[j]]}],{j,1,p}];
      Product[l[[j]]^r[[j]],{j,1,p}]*Sum[P[z][[j]]*Exp[-l[[j]]*z],{j,1,p}]
      ]  ]

GIGcdf[r_, li_, z_] :=
  Module[{p,l,c},
    If[Count[r,_Integer]==Length[r]&&And@@Positive[r]&&And@@Positive[li]&&
                                                    Length[r]==Length[li],
      p = Length[r];
      l = Rationalize[li,0];
      c = Makec[r,l,p];
      P[w_]:=Table[Sum[c[[j]][[k]]*(k-1)!*Sum[w^i/(i!*l[[j]]^(k-i)),{i,0,k-1}],
                                                    {k,1,r[[j]]}],{j,1,p}];
      1-Product[l[[j]]^r[[j]],{j,1,p}]*Sum[P[z][[j]]*Exp[-l[[j]]*z],{j,1,p}]
      ]  ]

Makec[r_, l_, p_] := Module[{c},
    c = Table[Table[1,{j,1,Max[r]}],{i,1,p}];
    Table[c=ReplacePart[c,(Product[(l[[j]]-l[[i]])^(-r[[j]]),{j,1,i-1}]*
                   Product[(l[[j]]-l[[i]])^(-r[[j]]),{j,i+1,p}])/(r[[i]]-1)!,
                                                       {i,r[[i]]}],{i,1,p}];
    Table[Table[c=ReplacePart[c,Sum[((r[[i]]-k+j-1)!*(Sum[r[[h]]/(l[[i]]-l[[h]])^j,
                   {h,1,i-1}]+Sum[r[[h]]/(l[[i]]-l[[h]])^j,{h,i+1,p}])*
              c[[i]][[r[[i]]-(k-j)]])/(r[[i]]-k-1)!,{j,1,k}]/k,{i,r[[i]]-k}],
                                                    {k,1,r[[i]]-1}],{i,1,p}];
    c
      ]
```

Figure D.2 – Mathematica modules `GIGpdf` and `GIGcdf` called by `PDFL` and `CDFL` in Figure D.1.

An example of a command which may be used to obtain the 0.05 exact quantile of Λ_2 in (15) for $p = 6$ and $N = 50$ would be

```
FindRoot[CDFL[50,6,x]==5/100,{x,0.4},WorkingPrecision->30]
```

since the first argument of both `PDFL` and `CDFL` is the sample size, the second is the number of variables and the third is the running variable for the function. For larger values of p we would have to adequately increase the value for `WorkingPrecision`.

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