

**A Strategic Market Game Approach for the Private Provision of
Public Goods¹**

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Abstract. Bergstrom, Blume and Varian (1986) provides an elegant game-theoretic model of an economy with one private good and one public good. Strategies of players consist of voluntary contributions of the private good to public good production. Without relying on first order conditions, the authors demonstrate existence of Nash equilibrium and an extension of Warr's neutrality result – any redistribution of endowment that left the set of contributors unchanged would induce a new equilibrium with the same total public good provision. The assumption of one-private good greatly facilitates the results. We provide analogues of the Bergstrom, Blume and Varian results in a model allowing multiple private and public goods. In addition, we relate the strategic market game equilibrium to the private provision of equilibrium of Villanaci and Zenginobuz (2005), which provides a counter-part to the Walrasian equilibrium for a public goods economy. Our techniques follow those of Dubey and Geanakoplos (2003), which itself grows out of the seminal work of Shapley and Shubik (1977). Our approach also incorporates, into the strategic market game literature, economies with production, not previously treated and, as a by-product, establishes a new existence of private-provision equilibrium.

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1 Strategic market games and public goods

One of the most important papers on public good provision is Bergstrom, Blume and Varian (1986), BBV. This paper brought strategic behavior in public provision to the forefront of public economic theory. In contrast to personalized prices for public goods, as in Samuelson (1954) and Foley (1970) for example, BBV treats a game-theoretic model in which the strategies of the players are their own voluntary contributions to public good provision. The BBV model focuses on a situation with one private good that can be consumed or contributed to public good provision. In addition to existence of equilibrium, BBV demonstrate conditions under which Warr's neutrality result – redistributions of endowments do not change the equilibrium allocation of private goods and the total amount of public good provided – continues to hold.⁴

The elegant model of BBV raises a number of challenges, including the development of a strategic model for the analysis of voluntary contributions equilibrium in situations with multiple private goods and with production of public goods. This challenge motivates the current paper. In the context of an economy with finite numbers of agents (consumers and firms), we first introduce the concept of a private-provision equilibrium, due to Villanacci and Zenginobuz (2005), VZ. In private-provision equilibrium, agents take prices for private and public goods as given, firms maximize profits, and subject to their budget constraints, consumers choose their private goods consumptions and the amounts of their contributions to the provision of the public good. We first extend the concept of private provision equilibrium to treat both multiple public and private goods. We then develop a continuum representation of the economy. The equilibrium outcomes of the associated economy with a continuum of agents coincide with those of the original economy with a finite number of agents. This aspect is novel and is, in part, due to our representation of utility functions of consumers in a continuum. In our continuum model, with regard to private goods, players are negligible while, with regard to public goods, consumers are in the same

⁴There are numerous precursors to the BBV model and results; see their paper for references. Many other authors have studied existence of equilibrium and Warr's neutrality result in a variety of contexts; see, for example, Kemp (1984), Itaya, de Meza and Myles (2002), Cornes and Itaya (2010), Silvestre (2012), Allouch (2012) and others. A recent contribution to the literature on existence is provided in Florenzano (2009), which highlights the similarities with existence of price-taking equilibrium in other contexts and, like this paper, allows multiple public and private goods.

situations as the consumers in the finite economy.

We next introduce a strategic model with a continuum of players. Our strategic model is an adaptation of a Shapley-Shubik market game, as developed by Dubey and Geanakoplos (2003). In these papers, trading is carried out at specialized trading posts – each post specializes in the trade of one private good. We add provision posts for contributions of public goods. Roughly, we show that for any finite economy there is an associated market game with a continuum of players (consumers and firms) with the property that the strategic Nash equilibria of the market game induce private provision equilibria of the finite economy. The intuitive meaning of our result is that in a “large” (but finite) economy, Nash equilibria approximate private provision equilibria. We also demonstrate a neutrality result.

Let us provide more background and motivation for our work and also some additional discussion. The private provision equilibrium model of VZ treats a general equilibrium model with multiple private good and one public good. Their private provision equilibrium concept is in the tradition of classic general equilibrium models in that agents⁵ take prices as given and maximize their payoffs – consumers maximize their utilities and firms maximize their profits. An innovative feature of the model of VZ is that agents purchase the public good at its per unit supply price, rather than at personalized prices as in Samuelson (1954) and Foley (1970) and choose the amount of the public good they purchase. The sum, over all agents, of the amounts of public good purchased is the total amount of the public good provided to the consumers in the economy. The model of VZ is important and interesting. It provides an analogue, in the tradition of Walras, of the model and voluntary contributions equilibrium of BBV. The model and equilibrium of VZ, however, has not until now had foundations in strategic game-theoretic equilibrium.⁶

In this paper we provide strategic foundations for private provision equili-

⁵When considering general equilibrium models and, in particular, the model of VZ we will refer to the members of the economy as “agents” while when we consider strategic models we will refer to the members of the economy as “players”. We use the terms “consumers” and “firms” in both situations.

⁶Rephrasing some remarks in Dubey and Genakoplos (2003, p. 392), “the most salient feature of the game theoretic-theoretic approach is that .. no matter what strategies agents choose a feasible outcome is always engendered. In Walrasian analysis (and in the analysis of VZ) we are left in the dark as to what happens out equilibrium.” (Insertions in parentheses are ours.)

brium. Our approach owes much to Dubey and Geanakoplos (2003), DG. Recall that DG demonstrates that Walrasian equilibria of a private goods economy are the limits of Nash equilibria of a sequence of strategic games. The DG paper is deep and insightful. It also provides an excellent discussion of the related literature on the strategic foundations of Walrasian equilibrium. We briefly described their model and approach. DG begin with a private goods exchange economy, a finite set of households, and a Walrasian equilibrium. They then develop a representation of the model with a continuum of players of a finite number of types. In the continuum representation, there is the same measure of all players of each type and all players of the same type are identical (and identical to one player in the finite economy). The DG continuum model is a variant of the Shapley-Shubik strategic trading-post game. There is a “trading post” for each commodity. Each player delivers his entire endowment of each commodity to the trading post designated for that commodity. Money can be borrowed to buy commodities and then paid back when the player receives monetary payments for his endowment. Money itself has no intrinsic value but there is a penalty for default if the value of the player’s endowment falls short of the value of his purchases. Also, there is a bound M on the amount of money that a player may borrow. Letting M go to infinity creates a sequence of games. Each game in the sequence has a Nash equilibrium. A limit of the equilibrium outcomes generates a Walrasian equilibrium for the initial economy.⁷ In this way, DG provides strategic foundations for the Walrasian equilibrium.

As noted, our strategic game approach follows that of DG but with the addition of “provision posts,” to which consumers can make contributions of money

⁷To motivate the need for a bound on money, recall that in their seminal paper, Shapley-Shubik take one commodity as numeraire. Each consumer chooses the amounts of this commodity to allocate to purchasing private goods. For existence of equilibrium, it is necessary to bound the strategy set of each consumer by her endowment of the numeraire commodity. In this case, with bounds on strategy sets, it is not necessarily true that a Nash equilibrium of the game is a Walrasian equilibrium.

Subsequent and related works consider fiat inside money. In these models, a bound in the amount of money that each consumer can borrow restricts admissible trades and, again, an equilibrium in the game cannot be a Walrasian equilibrium. DG consider a sequence of truncated games and show that the limit of a sequence of equilibria for the games is a Walrasian equilibrium.

In our model, the need to impose a bound on money and also to consider limits of equilibria is similar to that of DG – first, to obtain existence of Nash equilibrium and then to relate Nash equilibria of the strategic games to Walrasian equilibrium in the finite economy.

to provide the public good associated with each provision post. As in DG, consumers take their endowments of private goods to trading posts and can spend money on both private and public goods at the each post. The strategy of a consumer is the amount of money that she will spend (or contribute) to each post. The strategy of a firm states the amount that the firm will spend on each input. Prices at each post are determined by the total money spent at the post divided by the number of units of the commodity provided, either the aggregate endowment of that commodity or the aggregate amount of the public good provided by firms. With an additional condition bounding the numbers of units of produced commodities away from zero, we establish existence of Nash equilibrium. We show that as the bound on the numbers of units of produced goods goes to zero and M is allowed to go to infinity, there is a sequence of Nash equilibria with corresponding allocations and prices that converge to a private provision equilibrium for the finite economy.

One of the striking features of the BBV paper is their neutrality result, which generalizes those of Warr (1983) and Kemp (1984). BBV does not rely on first order conditions, but instead obtain their results using properties resulting from optimization by individual agents. In BBV, with one private good and one public good, the authors consider perturbations of the endowments of the consumers. They prove that an equilibrium for the economy generates an equilibrium for the perturbed economy in which all consumers have the same private goods allocation and the total public good contribution is unchanged, provided that the perturbation does not change the total amount of endowment of the economy and every consumer can afford his initial equilibrium consumption of private goods. We show that an analogous result holds for our model.

Our work has some quite novel features motivated by the complexity of issues involved. First, it is possible to model finite private goods exchange economies as market games with the same, finite set of players. But the complexities are considerable; we refer the reader to the excellent discussion of Dubey and Geanakoplos (2003). For private-goods price-taking to be close to fully rational and strategic behavior, we must have an economy with a large number of participants. But for a close approximation, as argued by a number of authors, we can use a continuum economy. In interpretation of our work we thus assume that having a continuum of firms well approximates a situation with a large but finite number of firms. For (pure) public goods, however, we cannot directly well approximate an economy or game with a finite number of players by a continuum

model; in such models, the individual consumer is negligible and his contribution to public good provision is negligible. (A first paper making this point is Muench, 1972). It is well known, however, and will be illustrated by an example, that in sequences of growing economies with a public good, the amount of public good may converge to a strictly positive amount. Thus, we do not treat a “continuum limit” model; we *represent* a finite game as a game with a continuum of players. Our continuum game representation of the finite model is “as if” there is a continuum of firms and consumers, insofar as private goods are involved but consumers, the contributors to public good provision, are influenced by their own public good provision and the average contributions of consumers of other types. (See the example in Section 4 and our concluding section). That is, in the continuum it is as if an individual is unaffected by the contributions of others of her own type or views herself as a representative member of her type. Thus, even though the individual consumer is in the same situation as in the finite economy with regard to the total amount of public good consumed, she is strategically negligible. We note that in the case of one-private-good our game-theoretic framework is equivalent to that of BBV.

As a by-product of our results, we obtain existence of the private provision equilibrium with both multiple public goods and private goods, thus extending the Villanacci and Zenginobuz result for the case of constant returns to scale in production of public goods.

The remainder of the paper is organized as follows. In Section 2 and 3, we present the model and the corresponding market equilibrium. In Section 4, we state some remarks with respect to large economies and public goods that might be helpful to a better understanding of the strategic market approach provided in this work. In Section 5, we define a game for the private provision of public goods and then, in Section 6, we show existence of Nash equilibrium for the games we consider. In Section 7, we obtain our main result which states that the limit of a sequence of Nash equilibria results in a market equilibrium. In Section 8 we provide a neutrality result. Section 9 concludes the paper. In a final Appendix, we present the proofs of all the results stated in this paper. But before leaving this introduction we comment further on the literature on strategic market games to further relate our work to the literature and to indicate future directions for research.

Remarks on selected literature on strategic market games:

1. A question that immediately arises in the strategic market game context is whether prices are positive. This question was addressed by Peck, Shell and Spear (1992) who demonstrate conditions on a private-goods economy under which there are strictly positive equilibrium bids for all goods (and provide an in-depth study of the model).⁸ We cannot establish such a result and do not aim to do so, given that we wish to allow situations where some consumers do not contribute to public good provision and we allow production, with the possibility that some public goods are not produced. Like our paper, Peck, Shell and Spear consider an ‘inside’ or fiat money, representing the private debt of the consumers with default penalties but, unlike the situation in our continuum game model, there is no bound on consumer debt. We require such a bound; otherwise consumer demands would be unlimited and, as DG, we wish to demonstrate that, with many players, price-taking equilibrium outcomes arise as outcomes of strategic behavior.⁹

2. Another important question is the convergence of equilibrium outcomes of finite economies to the continuum model. For private-goods economies, convergence of strategic market-game equilibrium to no-arbitrage equilibrium is demonstrated in Koutsougeras (2003a) and in Amir and Bloch (2009).¹⁰

3. For private-goods economies, Amir and Block (2009) demonstrate that when both goods are normal, prices increase with the number of buyers (holding the number of sellers constant). With strategic market games with public goods economies, it may be especially interesting to examine what happens to the set of contributors as a (finite) set of players increases in size. This question has already been examined in the literature in a path-breaking paper, Andreoni (2002), on private provision equilibrium; we conjecture that analogues of Andreoni’s (2002) results will hold for strategic market games with public goods and finite but growing player sets.¹¹

⁸Peck, Shell and Spear (1992) also treat existence, structure and dimension of the manifold of Nash equilibrium allocations in a strategic market game

⁹The treatment of money in a strategic market game has been a subject of intense debate. We prefer to have fiat money as we believe it is a better fit for the intuitive notions of a public goods economy underlying our model. We thank Gael Giraud, Hubert Kempf, and Herakles Polemarchakis for stressing this point. See Gael (2003) for some discussion of this and relevant references.

¹⁰See also Koutsougeras (2003b).

¹¹Mention of “Andreoni” recalls his theory of warm-glow giving (Andreoni 1990). We con-

4. But will players, over time, learn to play the equilibrium of a strategic market game in the presence of public goods? Brangewitz and Giraud (2011) address this issue for a very general model of a private-goods economy. It may be interesting to consider this question in a model with public goods but sufficiently specific to allow comparative statics of interest to public economic theory.

Finally, we refer the reader to Giraud (2003) for a recent review of the literature on strategic market games.

2 The model

We consider an economy \mathcal{E} with a finite number L of private goods and a finite number K of public goods. There is a set \mathcal{N} of N consumers who consume private goods and collectively consume public goods. Each consumer $i \in \mathcal{N} = \{1, \dots, N\}$ is characterized by her endowment of private goods $e_i \in \mathbb{R}_{++}^L$ and by her preference relation over commodity space \mathbb{R}_+^{L+K} . Preferences are represented by a continuous, concave and monotone-increasing utility function $U_i : \mathbb{R}_{++}^{L+K} \rightarrow \mathbb{R}_+$. Define $e = \sum_{i=1}^N e_i$.

There are K firms that produce public goods. A firm $k \in \{1, \dots, K\}$ is characterized by a production function $F_k : \mathbb{R}_+^L \rightarrow \mathbb{R}_+$ that converts private goods into public good k . We assume that each F_k is continuous, concave, and exhibits constant returns to scale. Each consumer $i \in \mathcal{N}$ owns a share $\delta_i^k \geq 0$ of the firm k 's profit and $\sum_{i=1}^N \delta_i^k = 1$ for each k .

3 Private provision equilibrium

A *price system* is a vector $(p, q) \in \mathbb{R}_+^{L+K}$, where $p = (p^\ell, \ell = 1, \dots, L)$ denotes the vector of prices for the L private commodities and $q = (q^k, k = 1, \dots, K)$ denotes the vector of prices for the K public goods.

Given a price system $(p, q) \in \mathbb{R}_+^{L+K}$ and profits Π_k for each firm k , consumers choose private goods consumption and voluntary contributions to public good provision. Each consumer takes as given the contributions of the other consumers. That is, given a vector $(g_j, j \in \mathcal{N}, j \neq i)$ of voluntary contributions,

jecture that our results will continue to hold when consumers preferences are defined over both the levels of public goods provided and their own contributions.

each consumer i solves the problem:

$$\max_{(x, \varrho) \in \mathbb{R}_+^L \times \mathbb{R}_+^K} U_i(x, g_{-i} + \varrho)$$

$$\text{such that } p \cdot x + q \cdot \varrho \leq p \cdot e_i + \sum_{k=1}^K \delta_i^k \Pi_k,$$

where $g_{-i} = \sum_{j \neq i} g_j$.

Definition: A *private provision equilibrium* for the economy \mathcal{E} is a price system (p, q) , a vector of inputs $y_k \in \mathbb{R}_+^L$ for every firm k , a private consumption allocation $(x_i, i = 1, \dots, N)$ and an allocation of public goods $\sum_{i=1}^N g_i = (g^k, k = 1, \dots, K)$ such that,

- (i) (x_i, g_i) solves the problem of consumer i for every $i \in \mathcal{N}$.
- (ii) y_k maximizes firm k ' profit, for every k .
- (iii) $\sum_{i=1}^N x_i + \sum_{k=1}^K y_k \leq \sum_{i=1}^N e_i$.
- (iv) $g^k \leq F_k(y_k)$ for every public good k .

4 Large economies and public goods

Roughly, given a finite economy, we formulate a continuum representation of the economy in which the actions of each consumer and each firm are negligible from the viewpoint of other agents but with the property that private goods provision equilibria for the continuum economy generate a private goods equilibrium for the finite economy. In our continuum representation, each consumer's utility function depends on her own consumption of private goods, her own provision of public goods and the average contribution of consumers of each of the other types.

Continuum economies with only private goods are well studied. The utility function of a consumer in the continuum depends only on her own consumption and the consumption bundle of a consumer depends only on her own purchases. In the presence of (pure) public goods, difficulties arise. The contribution of a consumer towards public good provision has a non-negligible effect on the consumer, in that it costs him private resources, but only a negligible effect

on the total amounts produced of public goods. Thus, the consumer optimizes by contributing zero to public good provision and, in an equilibrium, no public goods are produced.¹² Yet there are theoretical examples – one follows – where, as a sequence of economies becomes large, the amount of public good provided does not converge to zero but instead to some positive, finite amount.¹³

Example. Let us consider an economy with two agents (1 and 2) that consume one private good and one public good. Both consumers has the same utility function $U(x, G) = xG$, where x denotes the private commodity and G is the amount of public good. Each consumer owns initially one unit of the private good. The public good is produced via the production function $F(x) = x$. The equilibrium prices are given by $p = q = 1$, where p is the price for the private commodity and q is the price for the public good. The private provision equilibrium allocation for this economy is $x_1 = x_2 = \frac{2}{3}$, $g_1 = g_2 = \frac{1}{3}$ and $G = g_1 + g_2 = \frac{2}{3}$. Now if we consider the r -replica economy, with r agents of type 1 and r consumers of type 2, then the private provision equilibrium is the following, $p = q = 1$ and $x_{1j} = x_{2j} = \frac{2r}{2r+1}$, and $g_{1j} = g_{2j} = \frac{1}{2r+1}$ for every $j = 1, \dots, r$, which leads to $G = \frac{2r}{2r+1}$. Observe that the private provision of the public good provided by each consumer converges to zero when the number consumers, $2r$, converges to infinite. In spite of this, every consumer is better off when the economy is enlarged.

It is easy to show that if the utility function for each agent ij in the r -replica economy is $U_{ij}^r(x, g_{ij}, g_{-i}) = x(g_{ij} + g_{-i})$, where $rg_{-1} = \sum_{j=1}^r g_{2j}$ and $rg_{-2} = \sum_{j=1}^r g_{1j}$, then the private provision equilibrium is given by, $p = q = 1$, $x_{1j} = x_{2j} = \frac{2}{3}$, $g_{1j} = g_{2j} = \frac{1}{3}$ and then, $g_{1j} + g_{-1} = g_{2j} + g_{-2} = \frac{2}{3}$ for every j . ♦

Given the finite economy \mathcal{E} , let us consider an associated economy \mathcal{E}_C with a continuum of consumers represented by the real interval $C = [0, N]$ and a continuum of firms represented by the real interval $I = [0, K]$, both endowed with the Lebesgue measure μ . Each consumer i in the economy \mathcal{E} is represented in \mathcal{E}_c by the real interval $C_i = [i - 1, i)$ if $i \neq N$ and consumer N is represented by $C_N = [N - 1, N]$. Each firm k in \mathcal{E} is represented in \mathcal{E}_c by $I_k = [k - 1, k)$ if $k \neq K$ and firm K is represented $I_K = [K - 1, K]$. Every firm $h \in I_k$, has the same production function $F_k : \mathbb{R}_+^L \rightarrow \mathbb{R}_+$.

¹²Since Muench (1972), various approaches have been proposed to treat this problem. Muench introduces a distinction between “macro” quantities and “micro” quantities.

¹³Similar examples appear elsewhere; see, for example, Andreoni (1988).

Each consumer $t \in C_i$ has endowment $e_t = e_i$ and preference relation on the consumption of private commodities and public goods represented by the utility function V_t defined below.

In order to define utility functions V_t we require some notation. Let $g : [0, N] \rightarrow \mathbb{R}_+^K$ be a function which specifies a private provision of public goods $g(t)$ for every consumer t . We write

$$g_i = \int_{C_i} g(t) d\mu(t)$$

and

$$g_{-i} = \int_{C \setminus C_i} g(t) d\mu(t) = \sum_{j \neq i} g_j.$$

Having done this, the utility function of an agent $t \in I_i$ is given by

$$V_t(x, g_{-i}, g(t)) = U_i(x, g_{-i} + g(t)).$$

Each consumer $t \in C_i$ owns the share $\delta_t^k = \delta_i^k$ of the profits of the firms of type k . We will refer to individuals in C_i as consumers of type i .

At this point, the definition of a private provision equilibrium for the continuum economy is clear – agents maximize given prices and feasibility must be satisfied – so we do not provide a formal statement. Since the utility functions are concave it is easy to show that the following proposition holds:

Proposition 4.1 *A private provision equilibrium for the finite economy induces an equilibrium for the continuum economy and the converse.*

- (i) If prices (p, q) , input bundles $(y_k, k = 1, \dots, K)$, and allocations $(x_i, g_i, i = 1, \dots, N)$ constitute a private provision equilibrium for the economy \mathcal{E} then (p, q, y, x, g) is a private provision equilibrium for the economy \mathcal{E}_C , where $x(t) = x_i$ and $g(t) = g_i$ for every $t \in C_i$ and $y(h) = y_k$ for every $h \in I_k$.
- (ii) Reciprocally, if (p, q) , the μ -integrable function of inputs y and the allocations $(x(t), g(t), t \in C)$ constitute a private provision equilibrium for the economy \mathcal{E}_C , with $g(t) = g_i$ for every $t \in C_i$, then $(p, q, y_k, x_i, g_i, i = 1, \dots, N, k = 1, \dots, K)$ is a private provision equilibrium for the economy \mathcal{E} , where $x_i = \int_{C_i} x(t) d\mu(t)$ for every $i = 1, \dots, N$ and $y_k = \int_{I_k} y(h) d\mu(h)$ for every $k = 1, \dots, K$.

The relationships in Proposition 4.1 enable us to relate private provision equilibrium to the Nash equilibrium for the strategic market game defined in the next section.

5 A game for the private provision of public goods

In this section, given an economy \mathcal{E} we provide a strategic-form game where the players are the continuum of firms that produce the public goods and the continuum of consumers in the associated economy \mathcal{E}_C . The game models a situation where consumers take all their endowment of private goods to trading posts and borrow some amount of money, uniformly bounded by a constant $M > 0$, to spend on private goods and contribute to public good provision. If a consumer spends more than the value of her endowment, determined endogenously, she pays a penalty. The value of her endowment is determined by endogenously generated prices.

We now define the strategy choices available to the players and specify a price formation mechanism which defines a trading outcome. Note that, in contrast to the private provision equilibrium, a feasible outcome is determined for each strategy profile.

Following Shapley-Shubik (1977), each private commodity is traded at a trading post and, as in Dubey and Geanakoplos (2003), each player t surrenders her entire endowment of good ℓ to the ℓ^{th} post. Therefore the ℓ^{th} post receives the total endowment, in the continuum economy, of the ℓ^{th} good;

$$e^\ell = \int_C e_t^\ell d\mu(t) \quad (= \sum_{i=1}^N e_i^\ell).$$

Our strategic model also has a “provision post” for each of the K public goods. Consumers strategically choose amounts of fiat money to deliver to each post. The money delivered to each trading post ℓ is for purchase of private commodities and the amount delivered to the k^{th} provision posts is for provision of k^{th} public good. As in Dubey and Geanakoplos (2003), in order to trigger (or, in other words, to start) the market we assume that an external agent places 1 unit of fiat money in each of these $L + K$ posts.

A strategy for a consumer $t \in C$ is given by a vector $\theta_t \in \mathbb{R}_+^L$ that specifies an

amount of fiat money that she delivers to each trading post and a vector $\gamma_t \in \mathbb{R}_+^K$ that specifies the amount she delivers to each provision post. Given the bound M on total expenditure of a consumer, the *strategy set* for a consumer is given by

$$\mathcal{A}_M = \{s = (\theta, \gamma) \in \mathbb{R}_+^L \times \mathbb{R}_+^K : \text{such that } \sum_{\ell=1}^L \theta^\ell + \sum_{k=1}^K \gamma^k \leq M\}.$$

A *strategy for a firm* $h \in I$ is given by a vector ϕ specifying the amount of fiat money that the firm delivers to each one of the L trading posts to purchase private inputs in order produce the public commodities. Thus, the strategy set for a firm is

$$\mathcal{B}_M = \{\phi \in \mathbb{R}_+^L : 0 < \varepsilon \leq \sum_{\ell=1}^L \phi^\ell \leq MN + 1\}^{14}$$

A *strategy profile* is a triple $(\theta, \gamma, \phi) = ((\theta_t, \gamma_t)_{t \in C}, (\phi_h)_{h \in I})$ belonging to $\mathcal{A}_M^C \times \mathcal{B}_M^I$ and satisfying the property that the functions $(\theta, \gamma) : C \rightarrow \mathbb{R}_+^{L+K}$ and $\phi : I \rightarrow \mathbb{R}_+^L$ are μ -integrable.¹⁵

Given a strategy profile, $\xi = ((\theta_t, \gamma_t)_{t \in C}, \phi = (\phi_h)_{h \in I})$, prices for each private commodity $\ell \in \{1, \dots, L\}$ arise in each of the corresponding post according to the next rule:

$$p^\ell(\xi) = \frac{\vartheta^\ell + \varphi^\ell + 1}{e^\ell} > 0,$$

where $\vartheta^\ell = \int_C \theta_t^\ell d\mu(t)$ and $\varphi^\ell = \int_I \phi_h^\ell d\mu(h)$.

Let $p(\xi) := (p^\ell(\xi), \ell = 1, \dots, L) \in \mathbb{R}_+^L$. Given strategy profile ξ , let $x_t(\xi)$, defined by

$$x_t^\ell(\xi) = \frac{\theta_t^\ell}{p^\ell(\xi)}, \quad t \in C;$$

denote the commodity bundle assigned to the consumer $t \in C$ and let $y_h(\xi)$, defined by

$$y_h^\ell(\xi) = \frac{\phi_h^\ell}{p^\ell(\xi)}, \quad h \in I, \quad \ell = 1, \dots, L,$$

denote the bundle of inputs assigned to the firm $h \in I_k$ to produce the public good k .

¹⁴To obtain the private provision of public goods equilibrium as a limit of Nash equilibria, we will consider a sequence of games with M going to infinity and ε going to zero.

¹⁵We remark that we are going to deal only with symmetric strategy profiles, which allows us to avoid some measure-theoretic technicalities.

Firm $h \in I_k$ uses the bundle of inputs $y_h(\xi)$ to produce a level of public good k given by $G_h(\xi) = F_k(y_h(\xi))$. In order to define the price formation rule for the public goods, we need to avoid dividing by zero. Thus, given $\varepsilon > 0$ and a price system $p \gg 0$ for the private commodities, let us define the *minimum efficient production level* for firms that produces the public good k as follows: $m_\varepsilon^k(p) > 0$ is the maximum level of production of public good k that can be obtained with the vector of inputs given by $\frac{\phi^\ell}{p^\ell}$ such that $\sum_{\ell=1}^L \phi^\ell = \varepsilon$. That is, $m_\varepsilon^k(p) = F_k(\hat{y})$ where $\hat{y}^\ell = \frac{\hat{\phi}^\ell}{p^\ell}$ and $\hat{\phi}$ solves the following problem:

$$\max_{\phi} F_k \left(\frac{\phi^\ell}{p^\ell}, \ell = 1, \dots, L \right) \text{ such that } \sum_{\ell=1}^L \phi^\ell = \varepsilon.$$

In this way, the price for each public good k is defined as

$$q^k(\xi) = \begin{cases} \frac{\gamma^k + 1}{G^k(\xi)} & \text{if } G^k(\xi) \geq m_\varepsilon^k(p(\xi)) \\ \frac{\gamma^k + 1}{m_\varepsilon^k(p(\xi))} & \text{otherwise,} \end{cases}$$

$$\text{where } G^k(\xi) = \int_{I_k} G_h(\xi) d\mu(h) \text{ and } \gamma^k = \int_C \gamma_t^k d\mu(t).$$

Note that the price formation mechanism for public goods is well-defined and, for any $\varepsilon > 0$, gives incentives to firms to produce positive amounts of public goods, as we will show.

To complete the description of the game it remains to state the payoff functions for each player. Let Π_h denote the payoff function for a firm $h \in I$ and let Π_t denote the payoff function for a consumer $t \in C$. Given a strategy profile $\xi = (\theta, \gamma, \phi)$, define

$$\Pi_h(\xi) = q^k(\xi) F_k(y_h(\xi)) - \sum_{\ell=1}^L p^\ell(\xi) y_h^\ell(\xi), \text{ for each } h \in I_k.$$

In defining the payoff function Π_t for each consumer $t \in C$, we assume that fiat money has no utility. Given a strategy profile $\xi = (\theta, \gamma, \phi)$, the amount of public good k financed by player t is $g_t^k(\xi) = \frac{\gamma_t^k}{q^k(\xi)}$. Since a consumer t receives money from the sale of her endowment and her shares of the profits of firms, her net deficit is given by,

$$d_t(\xi) = d_t(\theta, \gamma, \phi) = \sum_{\ell=1}^L \theta_t^\ell + \sum_{k=1}^K \gamma_t^k - \sum_{\ell=1}^L p^\ell(\xi) e_t^\ell - \sum_{k=1}^K \delta_t^k \Pi_k(\xi),$$

where $\Pi_k(\xi) = \int_{I_k} \Pi_h(\xi) d\mu(h)$. We now define, for any strategy profile ξ , and every $t \in C_i$,

$$\Pi_t(\xi) = U_i(x_t(\xi), g_{-i}(\xi) + g_t(\xi)) - d_{t+}(\xi),$$

where $d_{t+}(\xi) = \max\{0, d_t(\xi)\}$ and $g_{-i}(\xi) = \int_{C \setminus C_i} g_t(\xi) d\mu(t)$.

Note that the use of the maximum to define the payoff functions for consumers means that, in spite of the fact that consumers do not ascribe utility to fiat money, they are penalized in the case of default, that is, if the consumer spends more than the value of her endowment.

Let us denote the game by $\mathcal{G}(\varepsilon, M)$. Given a strategy profile $\xi = ((\xi_t)_{t \in C}, (\xi_h)_{h \in I}) \in \mathcal{A}_M^C \times \mathcal{B}_M^I$, we denote by $\xi \setminus \xi'_h$ the strategy profile which coincides with ξ except for the firm h and where strategy ξ'_h replaces ξ_h . Denote by $\xi \setminus \xi'_t$ the profile which coincides with ξ except for consumer t and where strategy ξ'_t replaces ξ_t .

A *Nash equilibrium* for the game $\mathcal{G}(\varepsilon, M)$ is a strategy profile ξ such that $\Pi_t(\xi) \geq \Pi_t(\xi \setminus \xi'_t)$ for every $\xi'_t \in \mathcal{A}_M$ and almost all players $t \in [0, N]$ and $\Pi_h(\xi) \geq \Pi_h(\xi \setminus \xi'_h)$ for every $\xi'_h \in \mathcal{B}_M$ and for almost all firm $h \in [0, K]$.

6 A Nash equilibrium existence result

In this Section, we present an existence result for Nash equilibrium where every player of the same type selects the same strategy, that is, we show that the set of symmetric Nash equilibria for the game $\mathcal{G}(\varepsilon, M)$ is non-empty.

We say that a strategy profile (θ, γ, ϕ) is *symmetric* if (θ, γ) is constant in every C_i and ϕ is constant in every I_k , that is, $(\theta_t, \gamma_t) = (\theta_i, \gamma_i)$, for every $t \in C_i$ and $\phi_h = \phi_k$ for every $h \in I_k$. In other words, in any symmetric strategy profile, firms with the same technology select the same strategy and consumers of the same type select the same strategy. Thus, any symmetric strategy profile belongs to $\mathcal{A}_M^N \times \mathcal{B}_M^K$.

Theorem 6.1 *For every $0 < \varepsilon < M$ the set of symmetric Nash equilibria for the game $\mathcal{G}(\varepsilon, M)$ is non-empty.*

Lemma 6.1 *Let $\xi = (\theta, \gamma, \phi) \in \mathcal{A}_M^N \times \mathcal{B}_M^K$ be a symmetric Nash equilibrium for the game $\mathcal{G}(\varepsilon, M)$. Then $\Pi(\xi) = 0$ and $G^k(\xi) > 0$ for every public good k .*

Remark. The positivity of production levels of each public good is a result of our price formation rule for public goods, which is defined using the minimum efficient production level. This is crucial for our incorporation of production into the strategic market game approach.¹⁶

7 The main result

Let us consider a sequence of positive real numbers ε_M which converges to zero when M goes to infinity. In this section, we show that a private provision equilibrium for the economy \mathcal{E} can be obtained as the limit of a sequence of prices and allocations resulting from the sequence of symmetric Nash equilibria of the games $\mathcal{G}(\varepsilon_M, M)$ when M goes to infinity. For it, given a vector $a \in \mathbb{R}_+^L$, let $\|a\| \equiv \sum_{\ell=1}^L a^\ell$.

Theorem 7.1 *For each natural number M , let $\xi_M = (\theta_{M,t}, \gamma_{M,t}, \phi_{M,h}, t \in C, h \in I)$ be a symmetric Nash equilibrium for the game $\mathcal{G}(\varepsilon_M, M)$.*

Let $(p_M, q_M, x_M, g_M, y_M)$ be the corresponding sequence of prices and allocations defined by this sequence of Nash equilibria.

Then, there exists a subsequence of $\left(\frac{(p_M, q_M)}{\|p_M, q_M\|}, x_M, g_M, y_M\right)$ that converges to a price system (p, q) and an allocation (x, g, y) such that (p, q, x, g, y) is a private provision equilibrium for the economy \mathcal{E} . Neutrality

Let us consider the economy \mathcal{E} described in Section 2. We do not require constant returns to scale, however.

A redistribution of endowments is any allocation \hat{e} such that $\sum_{i=1}^N e_i = \sum_{i=1}^N \hat{e}_i$. Let $\mathcal{E}(\hat{e})$ denote the economy which coincides with \mathcal{E} except for the endowment is given by \hat{e} , a redistribution of e .

Lemma 7.1 *Let (p, q, x, g) be prices and allocations such that $p \cdot x_i + q \cdot g_i = p \cdot e_i$ for every consumer i . Consider a redistribution \hat{e} of endowments such that $p \cdot x_i \leq p \cdot \hat{e}_i$, for every i . Then, there exists an allocation of public goods \hat{g} such that $q \cdot \hat{g}_i = p \cdot (\hat{e}_i - x_i)$ for every consumer i and $\sum_{i=1}^N g_i = \sum_{i=1}^N \hat{g}_i$.*

¹⁶We thank Ali Khan and Rabah Amir for discussions pointing to the importance of this feature of our model.

Note that the redistribution in the above Theorem allows each consumer to afford the equilibrium bundle of private goods she is assigned in the initial allocation. This is important to the Lemma and to the following result.

Theorem 7.2 (*Neutrality*). *Let $(p^*, q^*, x_i^*, g_i^*, i = 1 \dots, N)$ be a private provision equilibrium for the economy \mathcal{E} . Let \hat{e} be a redistribution of endowments among contributing consumers for every public good, such that $p^* x_i^* \leq p^* \hat{e}_i$, for every consumer i . Then, there exists an allocation of public goods $\hat{g}_i, i = 1 \dots, n$, such that $(p^*, q^*, x_i^*, \hat{g}_i, i = 1 \dots, N)$, is a private provision equilibrium for the economy $\mathcal{E}(\hat{e})$ and $\sum_{i=1}^N \hat{g}_i = \sum_{i=1}^N g_i^*$.*

Consider the initial equilibrium $(p^*, q^*, x_i^*, g_i^*, i = 1 \dots, N)$. One candidate for an equilibrium after a redistribution satisfying the conditions of the Lemma is given by $(p^*, q^*, x_i^*, \hat{g}_i, i = 1 \dots, N)$, where $\sum_{i=1}^N \hat{g}_i = \sum_{i=1}^N g_i^*$. Note that in the proposed equilibrium, each consumer must still be maximizing. He can afford his original allocation of private goods and the total amount of public goods is the same. Also, the same equilibrium prices can hold in both the first and the second equilibrium. In the one-private-good case, this is essentially the same as the BBV neutrality result. BBV, however, consider “redistributions of income among contributing consumers such that no consumer loses more income than his original contribution,” while we consider redistributions with the property that no consumer loses more income (determined by the original equilibrium prices) than that required to buy his initial equilibrium bundle. In the one-private-goods case, these two sorts of redistributions are flip-sides of the same assumption. In the multiple-private-goods case it is more subtle.

Our restriction on redistributions is crucial. In Villanaci and Zenginobuz (2007), with many private goods and a strictly decreasing returns to scale production technology for the public good, relative price effects of redistribution have consequences. VZ (2007) shows that under exactly the same kind of redistribution as in BBV neutrality (of all equilibria) will not follow if one allows for relative price effects.¹⁷ They show that there exists redistributions of endowments that satisfy BBV’s requirements and at the same time decrease the overall public good level (or increase it if that is what is wanted). Change in the relative price is the key in obtaining their non-neutrality result. In contrast, our result shows that there exists an equilibrium that satisfies neutrality; we do not claim

¹⁷See also Villanaci and Zenginobuz (2012).

that *all* equilibria after redistribution satisfy neutrality; for such a claim to hold, at least some additional conditions would be required.

Our strategic market game results show not only that there are equilibria that satisfy neutrality but also that these equilibria can be approximated by the equilibria of strategic games with many players.

8 Conclusions

In summary, as the discussion and citations in Dubey and Geanakoplos makes clear, providing strategic foundations for the Walrasian equilibrium has been an important item on the research agenda of economics. Using the market-game approach, the current paper demonstrates that analogous foundations hold for the private provision equilibrium of Villanacci and Zenginobuz (2005) and the results of Bergstrom, Blume and Varian (1986) are approximated by strategic equilibrium of a fully general model of strategic equilibrium.

Appendix: Proofs.

Proof of Theorem 6.1. Let \mathcal{F}_h be a correspondence which associates to each symmetric strategy profile in $\mathcal{A}_M^N \times \mathcal{B}_M^K$ the best reply of the firm $h \in I$. That is, given the strategy profile $\xi = (\theta, \gamma, \phi) \in \mathcal{A}_M^N \times \mathcal{B}_M^K$

$$\mathcal{F}_h(\xi) = \arg \max_{\xi_h \in \mathcal{B}_M} \Pi_h(\xi \setminus \xi_h)$$

Note that, by symmetry, \mathcal{F}_h is the same for every firm $h \in I_k$ so, for simplicity, we denote \mathcal{F}_h by \mathcal{F}_k . By definition, $p(\xi) = p(\xi \setminus \xi_h)$ which allows us to obtain that $y_h(\xi \setminus \xi_h)$ is linear in ξ_h . Moreover, since F_k is concave for each k and $q(\xi) = q(\xi \setminus \xi_h)$ we can conclude that Π_h is concave in the firm h 's strategy. From the price formation rule, for each commodity ℓ it holds that $p^\ell(\xi) \geq 1/e^\ell$ and p is continuous in ξ . Moreover, the Maximum Theorem ensures that for every k the function m_ε^k is continuous in prices p , which implies that q is also continuous in ξ . Thus, the payoff function Π_h is continuous in the strategy profile ξ and \mathcal{B}_M is a non-empty convex and compact set. Consequently, \mathcal{F}_k takes non-empty and convex values and the Maximum Theorem allows us to conclude that the correspondence \mathcal{F}_k from $\mathcal{A}_M^N \times \mathcal{B}_M^K$ to \mathcal{B}_M is upper semi-continuous for every $k = 1, \dots, K$.

Let \mathcal{C}_t be a correspondence which associates to each symmetric strategy profile the best reply of the player $t \in C$. That is, given the strategy profile $\xi = (\theta, \gamma, \phi) \in \mathcal{A}_M^N \times \mathcal{B}_M^K$

$$\mathcal{C}_t(\xi) = \arg \max_{\xi_t \in \mathcal{A}_M} \Pi_t(\xi \setminus \xi_t)$$

Note that, by symmetry, \mathcal{C}_t is the same for every consumer $t \in C_i$ and we denote \mathcal{C}_t by \mathcal{C}_i . By definition, $p(\xi) = p(\xi \setminus \xi_t)$ which implies that $x_t(\xi \setminus (\theta_t, \gamma_t))$ is linear in θ_t and does not depend on γ_t . Moreover, since $q(\xi) = q(\xi \setminus \xi_t)$, we have that, for every k , $g_t^k(\xi \setminus (\theta_t, \gamma_t))$ is linear in γ_t^k and does not depend on θ_t (which follows from our assumption of a continuum of players). Since U_i is both concave and monotonic and $d_t(\xi \setminus \xi_t)$ is linear in ξ_t , we have that the payoff function Π_t is concave in the strategy selected by player t . Furthermore, \mathcal{A}_M is a non-empty, convex and compact set and, since the prices p and q are continuous in ξ and U_i is a continuous function, we can deduce that the payoff function Π_t is continuous in ξ . This implies that \mathcal{C}_i takes non-empty convex values and

the maximum theorem allows us to conclude that the correspondence \mathcal{C}_i , from $\mathcal{A}_M^N \times \mathcal{B}_M^K$ to \mathcal{A}_M , is upper semi-continuous for every $i = 1, \dots, N$.

Finally, let us consider the correspondence $\Gamma = (\mathcal{C}_1, \dots, \mathcal{C}_N, \mathcal{F}_1, \dots, \mathcal{F}_K)$. By Kakutani's theorem Γ has a fixed point, which actually is a symmetric Nash equilibrium.

Q.E.D.

Proof of Lemma 6.1. Since $\phi_h = \phi_k$ for every $h \in I_k$, we have that $\Pi_h(\xi) = \Pi_k(\xi)$ for every firm $h \in I_k$. Let us suppose that $\Pi_k(\xi) < 0$ (respectively $\Pi_k(\xi) > 0$) for some k . This implies that $\sum_{\ell=1}^L \phi_k^\ell > 1 > \varepsilon$ (respectively $\sum_{\ell=1}^L \phi_k^\ell < \sum_{i=1}^N \gamma_i^k + 1 \leq MN + 1$). Then we can take $\lambda < 1$ (respectively $\lambda > 1$) so that $\lambda \phi_k \in \mathcal{B}_M$ and $\Pi_k(\xi \setminus \lambda \phi_k) > \Pi_k(\xi)$ which is a contradiction with the fact that ξ is a Nash equilibrium.

To finish the proof, let us show that $F_k(y_k(\xi)) \geq m_\varepsilon^k(p(\xi))$ for every k . To obtain a contradiction, assume that $F_k(y_k(\xi)) < m_\varepsilon^k(p(\xi))$ for some k . Then, $\Pi_k(\xi) = \frac{\gamma^k + 1}{m_\varepsilon^k(p(\xi))} F_k(y_k(\xi)) - \sum_{\ell=1}^L \phi_k^\ell < \gamma^k + 1 - \sum_{\ell=1}^L \phi_k^\ell$. Consider a strategy $\hat{\phi}_k$ such that $m_\varepsilon^k(p(\xi)) = F_k(y_k(\xi \setminus \hat{\phi}_k))$. We have that $\Pi_k(\xi \setminus \hat{\phi}_k) = \gamma^k + 1 - \sum_{\ell=1}^L \hat{\phi}_k^\ell = \gamma^k + 1 - \varepsilon > \Pi_k(\xi)$. Therefore, if the symmetric strategy profile $\xi = (\theta, \gamma, \phi) \in \mathcal{A}_M^N \times \mathcal{B}_M^K$ is a Nash equilibrium then $G^k(\xi) \geq m_\varepsilon^k(p(\xi))$, for every k .

Q.E.D.

Proof of Theorem 7.1. Since $\xi_M = ((\theta_{M,t}, \gamma_{M,t}, t \in C), \phi_{M,h}, h \in I)$ is a symmetric Nash equilibrium for the game $\mathcal{G}(\varepsilon_M, M)$, we have $\theta_{M,t} = \theta_{M,i}$ for every $t \in C_i$ and every type i of consumers and $\phi_{M,h} = \phi_{M,k}$ for every $h \in I_k$. This equilibrium defines the prices $p_M = (p_M^\ell, \ell = 1, \dots, L)$ and $q_M = (q_M^k, k = 1, \dots, K)$ which leads to the allocation $x_M = (x_{M,i}, i = 1, \dots, N)$, contributions to public goods $g_M = (g_{M,i}, i = 1, \dots, N)$, inputs $y_{M,k}$ used in the production of the public good k and net deficits $(d_{M,i}, i = 1, \dots, N)$.

The definition of the game ensures that

$$\int_C x_{M,t} d\mu(t) + \int_I y_{M,h} d\mu(h) = \sum_{i=1}^N x_{M,i} + \sum_{k=1}^K y_{M,k} \leq e = \sum_{i=1}^N e_i = \int_C e_t d\mu(t).$$

Thus, the consumption bundles allocated to consumers x_M and the sequence of inputs y_M are bounded. Moreover, since $\sum_{i=1}^N g_{M,i}^k \leq F_k(y_{M,k}) \leq F_k(e)$, we have that each sequence $g_{M,i}^k$ is also bounded.

Note that if a player $t \in C_i$ selects the strategy $\theta = 0$ and $\gamma = 0$, then she has payoff $U_i(0, g_{-i}(\xi_M)) \geq U(0, 0)$, with $g_{-i}(\xi_M) = \sum_{j \neq i} g_{M,j}$. This implies that $U_i(e, F(e)) - d_{M,i+} \geq U_i(x_{M,i}, \sum_{i=1}^N g_{M,i}) - d_{M,i+} \geq U_i(0, 0)$ and, consequently, $d_{M,i+}$ is bounded from above by $U_i(e, F(e)) - U_i(0, 0)$.

For each M let us consider the sets of types of consumers defined as follows:

$$D_M = \{i \in \{1, \dots, N\} \text{ such that } d_{M,i} > 0\} \quad \text{and}$$

$$S_M = \{i \in \{1, \dots, N\} \text{ such that } d_{M,i} < 0\}.$$

That is, D_M is the subset of types agents who are in deficit and S_M is the set of agents who are in surplus. It trivially holds the next equality

$$\sum_{i=1}^N d_{M,i} = \sum_{i \in D_M} d_{M,i} + \sum_{i \in S_M} d_{M,i}.$$

By Lemma 5.1 equilibrium profits are null, that is, $\Pi_{M,k} = \Pi_k(\xi_M) = 0$ for every k and every M . Simple calculations then show that

$$0 = L + K + \sum_{i=1}^N d_{M,i} = L + K - \sum_{i \in S_M} -d_{M,i} + \sum_{i \in D_M} d_{M,i},$$

which implies that $\sum_{i \in S_M} -d_{M,i} = L + K + \sum_{i \in D_M} d_{M,i}$ is also bounded from above. Since $d_{M,i+}$ is bounded it follows that $-d_{M,i}$ is also bounded. Finally, we can conclude that $d_{M,i}$ is bounded.

Thus if we consider a sequence $(y_M, x_{M,i}, g_{M,i}, d_{M,i}, i = 1, \dots, N)_M$ with M going to infinity, there exists a converging subsequence with limit $(y, x_i, d_i, i = 1, \dots, N)$. Moreover, the sequence $\frac{(p_M, q_M)}{\|p_M, q_M\|}$ has also a convergent subsequence with limit (p, q) . We write $y_M \rightarrow y$, $x_{M,i} \rightarrow x_i$, $d_{M,i} \rightarrow d_i$, for each type i and $\frac{(p_M, q_M)}{\|p_M, q_M\|} \rightarrow (p, q)$.

For each M we have $\Pi_{M,k} = q_M^k F_k(y_{M,k}) - p_M y_{M,k}$. As we have already remarked, by Lemma 5.1, $\Pi_{M,k} = 0$. Then, $d_{M,i} = \sum_{\ell=1}^L \theta_{M,i}^\ell + \sum_{k=1}^K \gamma_{M,i}^k - p_M e_i$. Moreover, since $L + K > 0$ the set S_M is nonempty. Moreover, every consumer of type i in S_M must bid all the money that she can borrow. Otherwise, such a

consumer could increase the bidding in every private commodity, which entails a strict increase in the private consumption quantities of her bundle without incurring any penalty; in consequence her payoff will increase, which contradicts the the supposition of a symmetric Nash equilibrium. Then, since any agent of type $i \in S_M$ has surplus we have that $p_M e_i > M$, which implies that $\|p_M\| \rightarrow \infty$ when $M \rightarrow \infty$ and, in consequence, that $\|p_M, q_M\| \rightarrow \infty$ when $M \rightarrow \infty$

Now, we can write

$$\begin{aligned} \frac{d_{M,i}}{\|p_M, q_M\|} &= \frac{\sum_{\ell=1}^L \theta_{M,i}^\ell + \sum_{k=1}^K \gamma_{M,i}^k - \sum_{\ell=1}^L p_M^\ell e_i^\ell}{\|p_M, q_M\|} \\ &= \frac{\sum_{\ell=1}^L p_M^\ell x_{M,i}^\ell + q_M g_{M,i} - \sum_{\ell=1}^L p_M^\ell e_i^\ell}{\|p_M, q_M\|} \\ &= \frac{p_M}{\|p_M, q_M\|} (x_{M,i} - e_i) + \frac{q_M}{\|p_M, q_M\|} g_{M,i} \end{aligned}$$

Since $\|p_M\| \rightarrow \infty$ and $d_{M,i}$ is bounded for every type i , it follows that $\frac{p_M}{\|p_M, q_M\|} (x_{M,i} - e_i) + \frac{q_M}{\|p_M, q_M\|} g_{M,i} \rightarrow 0$, that is, $p(x_i - e_i) + qg_i = 0$ for all $i \in \{1, \dots, N\}$.

Let us show that p is non-zero. To obtain a contradiction, assume that $p = 0$, that is, $\frac{p_M^\ell}{\|p_M, q_M\|}$ converges to zero for every ℓ . This implies that there exists a public good k such that $\frac{q_M^k}{\|p_M\|}$ goes to ∞ when M increases. Consider a real number $\lambda > 0$ and let $\Lambda = (\lambda, \dots, \lambda) \in \mathbb{R}_{++}^L$. For each M let $\widehat{\phi}_M^\ell = \frac{\lambda p_M^\ell}{\|p_M\|}$. Note that $\widehat{\phi}_M^\ell \in \mathcal{B}_M$ for all M large enough. If any firm $h \in I_k$ deviates from ξ_M and chooses $\widehat{\phi}_M = (\widehat{\phi}_M^\ell, \ell = 1, \dots, L)$ the prices remain the same and the payoff that this firm gets is

$$\Pi_h(\xi_M \setminus \widehat{\phi}_M) = q_M^k F_k \left(y(\xi_M \setminus \widehat{\phi}_M) \right) - p_M \cdot y(\xi_M \setminus \widehat{\phi}_M) = \frac{q_M^k}{\|p_M\|} F_k(\Lambda) - \sum_{\ell=1}^L \frac{p_M^\ell}{\|p_M\|} \lambda.$$

We remark that $F_k(\Lambda) > 0$, $\sum_{\ell=1}^L \frac{p_M^\ell}{\|p_M\|} \lambda = \lambda$ and $\frac{q_M^k}{\|p_M\|}$ is an unbounded sequence.

Then $\Pi_h(\xi_M \setminus \widehat{\phi}_M)$ is strictly positive for all M large enough, in contradiction with the fact that ξ_M is Nash equilibrium for $\mathcal{G}(\varepsilon_M, M)$ for every M . Therefore, since $e_i \gg 0$, we have $p x_i + q g_i = p e_i > 0$.

Let us show that for every i we have $U_i(z, g_{-i} + g) \leq U_i(x_i, g_{-i} + g_i)$ for any (z, g) in the budget set $B_i(p, q)$, where

$$B_i(p, q) = \{(\alpha, \beta) \in \mathbb{R}_+^{L+1} \text{ such that } p\alpha + q\beta \leq pe_i\}.$$

To this end, let us take any real number $\lambda \in (0, 1)$ and a bundle $(z, g) \in B_i(p, q)$. Then, $\lambda pz + q\lambda g \leq \lambda pe_i = \lambda(px_i + qg_i) < px_i + qg_i$. This implies that, for all M sufficiently large, $\frac{p_M}{\|p_M, q_M\|} \lambda z + \frac{q_M}{\|p_M, q_M\|} \lambda g < \frac{p_M}{\|p_M, q_M\|} x_{M,i} + \frac{q_M}{\|p_M, q_M\|} g_{M,i}$ and thus $p_M \lambda z + q_M g < p_M x_{M,i} + q_M g_{M,i} \leq M$. Let us consider the strategy $\hat{\xi}_M = (\hat{\theta}_M, \hat{\gamma}_M)$ given by $\hat{\theta}_M^\ell = p_M^\ell \lambda z^\ell$ and $\hat{\gamma}_M^k = q_M^k \lambda g^k$.

Note that $\sum_{\ell=1}^L \hat{\theta}_M^\ell + \sum_{k=1}^K \hat{\gamma}_M^k = p_M \lambda z + q_M \lambda g$ and $(\hat{\theta}_M, \hat{\gamma}_M) \in \mathcal{A}_M$. Note also that the net deficit that any consumer $t \in C_i$ obtains by deviating and selecting $\hat{\xi}_t = \hat{\xi}_M$ is

$$d_t(\xi_M \setminus \hat{\xi}_t) = \left[\sum_{\ell=1}^L \hat{\theta}_M^\ell + \sum_{\ell=1}^L \hat{\gamma}_M^k - p_M e_i \right]_+ \leq \left[p_M x_{M,i} + \sum_{k=1}^K \hat{\gamma}_M^k - p_M e_i \right]_+ = d_{M,i} = d_t(\xi_M).$$

Therefore, since ξ_M is a Nash equilibrium, it holds that

$$U_i(x_{M,i}, g_{-i}(\xi_M) + g_{M,i}) \geq U_i(\lambda z, g_{-i}(\xi_M) + \lambda g).$$

Finally, passing to the limit and observing that the choice of $\lambda < 1$ was arbitrary, we conclude that $U_i(x_i, g_{-i} + g_i) \geq U_i(z, g_{-i} + g)$.

Recall that $\Pi_h(\xi_M) = q_M^k F_k(y_{M,k}) - p_M y_{M,k} = 0$ for every $h \in I_k$ and then $q^k F_k(y_k) - p y_k = 0$ for every $h \in I_k$. To finish the proof, it remains to show that the input vector y_k maximizes profits for every firm $h \in I_k$ at prices (p, q) . Assume that there exists $z \in \mathbb{R}_+^L$ such that $q^k F_k(z) - p z > 0$ for some k . Note that, since F_k is continuous and $p \neq 0$, without loss of generality we can consider $z \gg 0$. This implies that $\frac{q_M^k}{\|p_M, q_M\|} F_k(z) - \frac{p_M}{\|p_M, q_M\|} z > 0$, for all M large enough. On the other hand, since $p \neq 0$ and $z \in \mathbb{R}_{++}$ there exists real numbers \underline{a} and \bar{a} such that $0 < \underline{a} < \sum_{\ell=1}^L p^\ell z^\ell < \bar{a}$ and then the strategy $\hat{\xi}_h$, given by $\hat{\xi}_h^\ell = \frac{p_M^\ell z^\ell}{\|p_M, q_M\|}$, belongs to \mathcal{B}_M for all M large enough. Note that $y_h(\xi \setminus \hat{\xi}_h) = \frac{z}{\|p_M, q_M\|}$. Moreover, there exists \hat{M} such that $\Pi_h(\xi_M \setminus \hat{\xi}_h) = q_M^k F_k\left(\frac{z}{\|p_M, q_M\|}\right) - p_M \frac{z}{\|p_M, q_M\|} > \Pi_h(\xi_M) = 0$, for all $M \geq \hat{M}$, for all $h \in I_k$. This is in contradiction with the fact that ξ_M is a Nash equilibrium of the game $\mathcal{G}(\varepsilon_M, M)$ for every M .

Q.E.D.

Proof of Lemma 8.1. Let $\Delta e_i = \hat{e}_i - e_i$.

We will define three different sets of consumers. To which set a consumer i belongs depends on whether $p\Delta e_i$ is less than, greater than, or equal to zero. For each set, we develop an algorithm to assign new levels of public good contributions to each agent in the set. These assignments are made in such a way as to ensure that the total of assignments $\sum_{i=1}^N \hat{g}_i$ equals the total amounts of public goods given by the initial allocation, $\sum_{i=1}^N g_i$.

Set A. Let A denote the set of consumers i for whom $p\Delta e_i < 0$. Thus, A consists of those consumers for whom the values of endowments have decreased.

Given $i \in A$, denote by $\bar{k}(i)$ the lowest index number on public goods for which the change in the absolute value of endowment $|p\Delta e_i|$ is less than the total value of her contributions summed over all the public goods with lower index numbers; that is, $\bar{k}(i) = \min\{k : |p\Delta e_i| \leq \gamma_i^k := \sum_{h=1}^k q^h g_i^h\}$.

For every agent $i \in A$ let us define Δg_i as follows:

$$\Delta g_i^k = \begin{cases} -g_i^k & \text{if } k < \bar{k}(i) \\ \frac{p \cdot \Delta e_i + \gamma_i^{k-1}}{q^k} & \text{if } k = \bar{k}(i) \\ 0 & \text{otherwise.} \end{cases}$$

That is, for the k th public good, $k < \bar{k}(i)$, the i^{th} consumer's new assignment of public good contribution is equal to zero. For $k = \bar{k}(i)$ the i^{th} consumer's contribution is equal, in value, to the difference between the change in the value of her endowment and the amount that she initially spent on the public goods indexed $1, \dots, \bar{k}(i)$, and, for $k > \bar{k}(i)$ the i^{th} consumer's new assignment of public good contribution is equal to her initial assignment .

For each $i \in A$, define $\hat{g}_i = g_i + \Delta g_i$.

Set B. Now let B denote the set of consumers i for whom $p \cdot \Delta e_i > 0$; that is, B is the set of consumers for whom the value of endowment increases under the redistribution. To construct variations of the public goods contributions for consumers in B by induction let us write $B = \{b_1, \dots, b_n\}$.

For each public good k , let $\eta^k(b_1) = \sum_{i \in A} q^k \Delta g_i^k$ (the sum, over the members of A , of the values of the changes in public good k defined above) and let $\rho_1^k = \left| \sum_{h=1}^k \eta^h(b_1) \right|$ denote the absolute value of the sum of these changes.

Now, for the first consumer in the set B , b_1 , let $\underline{k}(b_1) = \min\{k : p \cdot \Delta e_{b_1} \leq \rho_1^k\}$, the lowest index number on public goods for which the change in the value of

endowment $(p\Delta e_{b_1})$ is (weakly) less than the amounts by which the values of the public goods contributions of consumers in A of good k have decreased.¹⁸ For $\underline{k}(b_1) = 1$, define

$$\Delta g_{b_1}^k = \begin{cases} \frac{p \cdot \Delta e_{b_1}}{q^1} & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

For $\underline{k}(b_1) > 1$, $\Delta g_{b_1}^k$ is defined as follows

$$\Delta g_{b_1}^k = \begin{cases} \frac{|\eta^k(b_1)|}{q^k} & \text{if } k < \underline{k}(b_1) \\ \frac{p \cdot \Delta e_{b_1} - \rho_1^{k-1}}{q^k} & \text{if } k = \underline{k}(b_1) \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $|B| > 1$.

Define $\eta^k(b_i) = \eta^k(b_{i-1}) + q^k \Delta g_{b_{i-1}}^k$. Let $\rho_i^k = \left| \sum_{h=1}^k \eta^h(b_i) \right|$. For every $i \in B$ let $\underline{k}(b_i) = \min\{k : p \cdot \Delta e_{b_i} \leq \rho_i^k\}$. The modification $\Delta g_{b_i}^k$ is defined as follows. If $\underline{k}(b_i) = 1$, define

$$\Delta g_{b_i}^k = \begin{cases} \frac{p \cdot \Delta e_{b_i}}{q^1} & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

Otherwise, define

$$\Delta g_{b_i}^k = \begin{cases} \frac{|\eta^k(b_i)|}{q^k} & \text{if } k < \underline{k}(b_i) \\ \frac{p \cdot \Delta e_{b_i} - \rho_i^{k-1}}{q^k} & \text{if } k = \underline{k}(b_i) \\ 0 & \text{otherwise.} \end{cases}$$

Finally, $\Delta g_i = 0$ if $p \cdot e_i = p \cdot \hat{e}_i$.

By construction $q \cdot \Delta g_i = p \cdot \Delta e_i$ and $\sum_{i=1}^N g_i = \sum_{i=1}^N \hat{g}_i$.

Q.E.D.

Proof of Theorem 8.1. Since the production functions for public goods exhibit constant returns to scale, we have zero profits at equilibrium. Moreover, for every consumer i the bundle (x_1^*, G^*) , where $G^* = \sum_{i=1}^N g_i^*$, solves the following individual problem:

$$\begin{aligned} & \max_{(x, G) \in \mathbf{R}_+^{L+\kappa}} U_i(x, G) \\ & \text{such that} \quad p^* \cdot x + q^* \cdot G \leq p^* \cdot e_i + q^* \cdot g_{-i}^* \\ & \quad \quad \quad G \geq g_{-i}^* \end{aligned}$$

¹⁸There is such a $\underline{k}(1)$ since $q \cdot \Delta g_i = p \cdot \Delta e_i$ for all $i \in A$ and then, for all $b_i \in B$,

$|\sum_{i \in A} q \cdot \Delta g_i| \geq p \cdot \Delta e_{b_i}$ because $|\sum_{i \in A} p \cdot \Delta e_i| = \sum_{b_i \in B} p \cdot \Delta e_{b_i}$.

where $g_{-i}^* = \sum_{j \neq i} g_j^*$.

By the previous Lemma we can take \hat{g} such that $q^* \cdot \hat{g}_i = p^* \cdot \hat{e}_i - p^* \cdot x^*$ for every i and $\sum_{i=1}^N \hat{g}_i = G^*$.

It remains to show that, for every i , the bundle (x_i^*, G^*) is a solution for the following problem:

$$\begin{aligned} \max_{(x,G) \in \mathbf{R}_+^{L+K}} \quad & U_i(x, G) \\ \text{such that} \quad & p^* \cdot x + q^* \cdot G \leq p^* \cdot \hat{e}_i + q^* \cdot \hat{g}_{-i} \\ & G \geq \hat{g}_{-i} \end{aligned}$$

where $\hat{g}_{-i} = \sum_{j \neq i} \hat{g}_j$.

Note that $p^* \cdot \hat{e}_i + q^* \cdot \hat{g}_{-i} = p^* \cdot e_i + q^* \cdot g_{-i}^*$.

Let us write $\hat{e}_i = e_i + \Delta e_i$.

If $p^* \cdot \Delta e_i < 0$ the budget set for consumer i becomes smaller and (x^*, G^*) belongs to it.

Consider the case $p^* \cdot \Delta e_i > 0$ and assume that there is a bundle (x, G) which is possible for agent i after redistribution of endowments and is preferred to (x_i^*, G^*) . Then, for every λ sufficiently close to 1, $\lambda(x_i^*, G^*) + (1 - \lambda)(x, G)$ is affordable for agent i before the redistribution of endowments and by convexity of preferences this bundle is also preferred to (x_i^*, G^*) , which is a contradiction.

Q.E.D.

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