Exact and Near-Exact Distribution of Positive Linear Combinations of Independent Gumbel Random Variables

Filipe J. Marques · Carlos A. Coelho · Miguel de Carvalho.

Abstract We show that the exact distribution of a positive linear combination of independent Gumbel random variables can be written as the sum of a linear combination of independent log Gamma distributions, and an independent shifted Generalized Integer Gamma distribution. Given the complexity of this exact distribution, we develop a near-exact distribution using a shifted Generalized Near-Integer Gamma distribution. Our numerical studies confirm the quality of this near-exact approximation as assessed by a proximity measure often used in related studies.

Keywords Generalized Integer Gamma distribution · Generalized Near-Integer Gamma distribution · Gumbel distribution · Near-exact distributions

Mathematics Subject Classification (2000) Primary 62E20 · 62H05; Secondary 62H10

1 Introduction

Despite the wide range of applications in which the distribution of the linear combination of independent Gumbel random variables may be useful, few results are available on this distribution. Nadarajah (2008) presents the exact distribution of the linear combination of p independent Gumbel random variables, using Fox H and Meijer G functions, but the computational investment required by these functions limits the practical usefulness of this result. Here we focus on positive linear combinations — which include convex linear combinations and sums as particular

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cases — and our aim is to develop approximations which are accurate and manageable. Our exact and near-exact approximations are based on the Generalized Integer Gamma (GIG) and Generalized Near-Integer Gamma (GNIG) distributions, which were introduced by Coelho (1998, 2004), and which have by now been used in a number of applications in multivariate analysis (Marques and Coelho, 2008; Coelho and Marques, 2010, 2011; Marques et al., 2011; Coelho et al., 2011). Details on the GIG and GNIG distributions are available in the Appendix A.

In Section 2.1 we show that the exact distribution of a positive linear combination of independent Gumbel random variables may be written as the sum of two independent random variables: the first corresponding to a linear combination of independent log Gamma random variables, and the second to a shifted Generalized Integer Gamma (SGIG) random variable. We should mention that our results may be readily applied to the product of powers of independent Weibull random variables, through a simple transformation, since if $X_j \sim \text{Gumbel}(\mu_j, \sigma_j)$, then $Y_j = \exp\{-X_j\} \sim \text{Weibull}(\exp\{\mu_j/\sigma_j\}, \sigma_j^{-1})$, thus implying that $\exp\{-\sum_{j=1}^{p} \alpha_j X_j\} = \prod_{i=1}^{p} Y_j^{\alpha_j}$, for any $\alpha_j \in \mathbb{R}$, for $j = 1, \ldots, p$. In Section 2.2 we develop a near-exact distribution, using a shifted Generalized Near-Integer Gamma (SGNIG) distribution with a precision parameter related with the depth of the SGNIG distribution; in the sequel we follow the convention that the last parameter in a shifted distribution is the shift parameter. Numerical studies are reported in Section 3.

2 Distribution Theory

2.1 Exact Distribution

Let $X_1, \ldots, X_p$ denote p independent Gumbel random variables, with location parameter $\mu_j \in \mathbb{R}$ and scale parameter $\sigma_j \in \mathbb{R}^+$, for $j = 1, \ldots, p$; that is

$$X_j \overset{\text{ind.}}{\sim} \text{Gumbel}(\mu_j, \sigma_j), \quad F_{X_j}(x) = \exp\{-\exp\{(x - \mu_j)/\sigma_j\}\}, \quad x \in \mathbb{R},$$

for $j = 1, \ldots, p$. The characteristic functions of X_j and $W = \sum_{j=1}^{p} \alpha_j X_j$, for $\alpha_j \in \mathbb{R}$, are respectively defined as

$$\Phi_{X_j}(t) = \Gamma(1 - \mathrm{i}t\sigma_j) \exp\{\mathrm{i}t\mu_j\}, \quad \Phi_W(t) = \prod_{j=1}^{p} \Gamma(1 - \mathrm{i}t\sigma_j\alpha_j) \exp\{\mathrm{i}t\mu_j\alpha_j\},$$

for $t \in \mathbb{R}$.

However, the exact distribution of the linear combination of independent Gumbel random variables does not have a manageable expression in practical terms. Although Nadarajah (2008) has presented the exact distribution of such a linear combination in terms of Fox H and Meijer G functions, since these functions are defined either by an unsolved integral or by an elaborate and slowly convergent infinite series their use in practical terms is not feasible. For this reason we propose in the next theorem a new characterization of the exact distribution of a linear combination of independent Gumbel random variables which will enable us to develop a sharp approximation in the next subsection. Below we use the notation $\mathbb{N}_*$ to denote the set $\{n \in \mathbb{N} : n \geq 2\}$.

Theorem 1 *Let* $X_j \overset{\text{ind.}}{\sim} \text{Gumbel}(\mu_j, \sigma_j)$*, with* $\mu_j \in \mathbb{R}$ *and* $\sigma_j \in \mathbb{R}^+$*. The exact characteristic function of* $W = \sum_{j=1}^{p} \alpha_j X_j$*, with* $\alpha_j \in \mathbb{R}$*,* $j = 1, \ldots, p$*, can be written as* $\Phi_W(t) = \Phi_{W_1}(t)\Phi_{W_2}(t)$*, where for any* $\gamma \in \mathbb{N}_*$ *and* $t \in \mathbb{R}$

$$\Phi_{W_1}(t) = \prod_{j=1}^{p} \frac{\Gamma(\gamma - \mathrm{i}t\sigma_j\alpha_j)}{\Gamma(\gamma)}, \tag{1}$$

and

$$\Phi_{W_2}(t) = \left\{ \prod_{j=1}^{p} \prod_{k=0}^{\gamma-2} \left(\frac{1+k}{\sigma_j\alpha_j}\right)\left(\frac{1+k}{\sigma_j\alpha_j} - \mathrm{i}t\right)^{-1} \right\} \exp\left\{ \mathrm{i}t \sum_{j=1}^{p} \mu_j\alpha_j \right\}. \tag{2}$$

Proof : Since we may write the characteristic function of W as

$$\begin{aligned}
\Phi_W(t) &= \prod_{j=1}^{p} \Gamma(1 - \mathrm{i}t\sigma_j\alpha_j)\, \exp\{\mathrm{i}t\mu_j\alpha_j\} \\
&= \left\{ \prod_{j=1}^{p} \frac{\Gamma(\gamma - \mathrm{i}t\sigma_j\alpha_j)}{\Gamma(\gamma)} \frac{\Gamma(\gamma)}{\Gamma(\gamma - \mathrm{i}t\sigma_j\alpha_j)} \frac{\Gamma(1 - \mathrm{i}t\sigma_j\alpha_j)}{\Gamma(1)} \right\} \exp\left\{ \mathrm{i}t \sum_{j=1}^{p} \mu_j\alpha_j \right\} \\
&= \prod_{j=1}^{p} \frac{\Gamma(\gamma - \mathrm{i}t\sigma_j\alpha_j)}{\Gamma(\gamma)} \left\{ \prod_{j=1}^{p} \prod_{k=0}^{\gamma-2} (1+k)\,(1+k - \mathrm{i}t\sigma_j\alpha_j)^{-1} \right\} \exp\left\{ \mathrm{i}t \sum_{j=1}^{p} \mu_j\alpha_j \right\} \\
&= \prod_{j=1}^{p} \frac{\Gamma(\gamma - \mathrm{i}t\sigma_j\alpha_j)}{\Gamma(\gamma)} \left\{ \prod_{j=1}^{p} \prod_{k=0}^{\gamma-2} \left(\frac{1+k}{\sigma_j\alpha_j}\right)\left(\frac{1+k}{\sigma_j\alpha_j} - \mathrm{i}t\right)^{-1} \right\} \exp\left\{ \mathrm{i}t \sum_{j=1}^{p} \mu_j\alpha_j \right\}.
\end{aligned}$$

the result in the theorem follows. □

If we consider now the case where $\alpha_j > 0$ ($j = 1, \ldots, p$), the exact distribution of $W = \sum_{j=1}^{p} \alpha_j X_j$ is the same as that of the sum of two independent random variables, W_1 and W_2, where

$$W_1 = -\sum_{j=1}^{p} \sigma_j\alpha_j \log Z_j, \quad Z_j \overset{\text{ind.}}{\sim} \text{Gamma}(\gamma, 1), \quad \gamma \in \mathbb{N}_*, \tag{3}$$

is a linear combination of p independent log Gamma random variables and W_2 is distributed according to a shifted sum of $p \times (\gamma - 1)$ independent Exponential distributions with parameters $(1+k)/(\beta_j\alpha_j)$, for $j = 1, \ldots, p$ and $k = 0, \ldots, \gamma - 2$, with shift parameter $\sum_{j=1}^{p} \mu_j\alpha_j$. If we sum the Exponential distributions with the same parameter, equation (2) can be written as

$$\Phi_{W_2}(t) = \left\{ \prod_{j=1}^{\ell} (\lambda_j)^{r_j} (\lambda_j - \mathrm{i}t)^{-r_j} \right\} \exp\left\{ \mathrm{i}t \sum_{j=1}^{p} \mu_j\alpha_j \right\}, \tag{4}$$

where ℓ is the number of Exponential distributions with different parameters, λ_j are the parameters of such Exponential distributions, and r_j is the number of Exponential distributions with the same rate parameter λ_j, for $j = 1, \ldots, p$. We have thus established the following corollary to Theorem 1.

Corollary 1 *Let* $X_j \overset{\text{ind.}}{\sim} \text{Gumbel}(\mu_j, \sigma_j)$, *with* $\mu_j \in \mathbb{R}$ *and* $\sigma_j \in \mathbb{R}^+$. *If* $W = \sum_{j=1}^{p} \alpha_j X_j$, *with* $\alpha_j \in \mathbb{R}^+$, $j = 1, \ldots, p$, *then it holds that* $W \overset{d}{=} W_1 + W_2$, *with* W_1 *as in* (3) *and*

$$W_2 \sim \text{SGIG}\left(r, \lambda, \ell, \sum_{j=1}^{p} \mu_j \alpha_j\right), \quad r = (r_1, \ldots, r_p), \quad \lambda = (\lambda_1, \ldots, \lambda_p). \tag{5}$$

In particular the Corollary above allows us to characterize the distribution of the sum of p independent Gumbel random variables. It is instructive to consider the case of the sum of p independent Gumbel random variables when $\sigma_j = \sigma$, $j = 1, \ldots, p$, for which simple expressions of the characteristic functions are readily available

$$\Phi_{W_1}(t) = \left(\frac{\Gamma(\gamma - \text{i}t\sigma)}{\Gamma(\gamma)}\right)^p, \quad \Phi_{W_2}(t) = \left\{\prod_{j=0}^{\gamma-2} (\lambda_j)^{r_j} (\lambda_j - \text{i}t)^{-r_j}\right\} \exp\left\{\text{i}t \sum_{j=1}^{p} \mu_j\right\}, \tag{6}$$

with $r_j = p$, $\lambda_j = (1 + j)/\sigma$, for $j = 0, \ldots, \gamma - 2$; this implies that in such case

$$W_2 \sim \text{SGIG}\left(p\mathbf{1}_{\gamma-1}^{\mathrm{T}}, \sigma^{-1}(1, \ldots, \gamma - 1), \gamma - 1, \sum_{j=1}^{p} \mu_j\right),$$

where $\mathbf{1}_{\gamma-1}$ denotes a $\gamma - 1$ vector of ones. The parameter γ is related with the depth of the SGIG distribution and it may be used as a precision parameter, since, as we will see in Section 3, for large values of γ we obtain better results for the near-exact distribution we propose below.

2.2 Near-Exact Distribution

Based on the characterization of the exact distribution of W in Corollary 1, we propose

$$\Phi_{W_1^\star}(t)\Phi_{W_2}(t)$$

as a near-exact characteristic function for W, where

$$\Phi_{W_1^\star}(t) = \left(\frac{l}{l - \text{i}t}\right)^\rho \exp\{\text{i}t\theta\}, \quad t \in \mathbb{R}, \tag{7}$$

is the characteristic function of $W_1^\star \sim \text{SGamma}(\rho, l, \theta)$ (see Appendix A) and replaces asymptotically $\Phi_{W_1}(t)$ in (1), for increasing values of γ.

The parameters ρ, l, and θ, will be determined as the numerical solution of the system of equations

$$\left.\frac{\partial^j \Phi_{W_1^\star}(t)}{\partial t^j}\right|_{t=0} = \left.\frac{\partial^j \Phi_{W_1}(t)}{\partial t^j}\right|_{t=0}, \quad j = 1, 2, 3\,. \tag{8}$$

The resulting near-exact distribution is established in the next theorem.

Theorem 2 *If we use as an asymptotic approximation of $\Phi_{W_1}(t)$ in* (1) *the characteristic function $\Phi_{W_1^\star}(t)$ in* (7)*, we obtain as near-exact distribution for $W = \sum_{j=1}^{p} \alpha_j X_j$, with $X_j \stackrel{\text{ind.}}{\sim} \text{Gumbel}(\mu_j, \sigma_j)$, and $\alpha_j \in \mathbb{R}^+$, for $j = 1, \ldots, p$, the distribution*

$$\text{SGNIG}\left(r^\star = (r_1, \ldots, r_\ell, \rho), \lambda^\star = (\lambda_1, \ldots, \lambda_\ell, l), \ell + 1, \theta + \sum_{j=1}^{p} \mu_j \alpha_j\right),$$

where r_j, λ_j, and ℓ are given in (4) *and ρ, l, and θ are obtained as the numerical solution of* (8)*.*

Proof : It is enough to note that

$$\begin{aligned}\Phi_{W_1^\star}(t)\Phi_{W_2}(t) &= \left(\frac{l}{l - \mathrm{i}t}\right)^{\rho} \exp\{\mathrm{i}t\theta\}\left\{\prod_{j=1}^{\ell} (\lambda_j)^{r_j} (\lambda_j - \mathrm{i}t)^{-r_j}\right\} \exp\left\{\mathrm{i}t\sum_{j=1}^{p} \mu_j \alpha_j\right\} \\ &= \left\{\prod_{j=1}^{\ell} (\lambda_j)^{r_j} (\lambda_j - \mathrm{i}t)^{-r_j}) l^{\rho} (l - \mathrm{i}t)^{-\rho}\right\} \exp\left\{\mathrm{i}t\left(\theta + \sum_{j=1}^{p} \mu_j \alpha_j\right)\right\}. \square\end{aligned}$$

It is again instructive to consider the particular case addressed in (6), that is when we consider the case of the sum of independent Gumbel random variables with the same scale parameter. In this case the system of equations in (8) has the solution

$$\rho = \frac{-4\nu_1^6 + 12\nu_1^4\nu_2 - 12\nu_1^2\nu_2^2 + 4\nu_2^3}{(2\nu_1^3 - 3\nu_1\nu_2 + \nu_3)^2}, \quad l = \frac{-2\nu_1^2 + 2\nu_2}{2\nu_1^3 - 3\nu_1\nu_2 + \nu_3},$$

and

$$\theta = \nu_1 - \frac{2\nu_1^4 - 4\nu_1^2\nu_2 + 2\nu_2^2}{\nu_1^3 - 3\nu_1\nu_2 + \nu_3}, \quad \nu_j = \mathrm{i}^{-j} \left.\frac{\partial^j \Phi_{W_1}(t)}{\partial t^j}\right|_{t=0} \quad j = 1, 2, 3\,.$$

Hence in this case we obtain the near-exact distribution

$$\text{SGNIG}\left(r^\star = (p\mathbf{1}_{\gamma-1}^{\mathrm{T}}, \rho), \lambda^\star = \sigma^{-1}(1, \ldots, \gamma - 1, l\sigma), \gamma, \theta + \sum_{j=1}^{p} \mu_j\right).$$

Modules for the implementation of the near-exact distribution proposed may be found in Appendix B.

3 Numerical Studies

To evaluate the quality of our near-exact approximation we consider the measure

$$\Delta = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left|\frac{\Phi_W(t) - \Phi^\star(t)}{t}\right| \mathrm{d}t\,,$$

which is known to verify the inequality

$$\sup_{w \in \mathbb{R}} |F_W(w) - F^\star(w)| \leq \Delta\,.$$

Table 1 Values of $\varDelta$ for scenarios I, II, and III

γ	Scenario I $(\mu_{\text{I}}, \sigma_{\text{I}}, \alpha_{\text{I}})$	Scenario II $(\mu_{\text{II}}, \sigma_{\text{II}}, \alpha_{\text{II}})$	Scenario III $(\mu_{\text{III}}, \sigma_{\text{III}}, \alpha_{\text{III}})$
4	1.4×10^{-4}	1.8×10^{-4}	3.4×10^{-4}
10	8.0×10^{-6}	1.0×10^{-5}	2.0×10^{-5}
15	2.3×10^{-6}	2.9×10^{-6}	5.8×10^{-6}
20	9.4×10^{-7}	1.2×10^{-6}	2.4×10^{-6}
50	5.8×10^{-8}	7.4×10^{-8}	1.5×10^{-7}
100	7.1×10^{-9}	9.1×10^{-9}	1.8×10^{-8}
500	5.6×10^{-11}	7.2×10^{-11}	1.4×10^{-10}

Here $\varPhi_W(t)$ and $\varPhi^\star(t)$ denote respectively the exact and approximate characteristic functions of W, and $F_W(w)$ and $F^\star(w)$ the corresponding cumulative distribution functions; details on this measure can be found in (Marques and Coelho, 2008; Coelho and Marques, 2010, 2011).

In Table 1 we present values of $\varDelta$ for different choices of μ_j, β_j, and α_j, according to the following scenarios:

—Scenario I: $\mu_{\text{I}} = (2, 3)$, $\sigma_{\text{I}} = (5, 6)$, and $\alpha_{\text{I}} = \mathbf{1}_2^{\text{T}}$;
—Scenario II: $\mu_{\text{II}} = (-4, -1, 2, 3)$, $\sigma_{\text{II}} = (0.1, 0.2, 0.3, 0.4)$, and $\alpha_{\text{II}} = (1, 2, 3, 4)$;
—Scenario III: $\mu_{\text{III}} = (-10, 10, 20, 30, 40)$, $\sigma_{\text{III}} = (1, 2, 3, 4, 5)$, and $\alpha_{\text{III}} = (1/2, 1, 3/4, 5, 1)$.

In Table 2 we report further numerical results on measure $\varDelta$ for sums of independent Gumbel random variables, all the same scale parameter, according to the following scenarios:

—Scenario 1: $\mu_1 = (2, 3)$, $\sigma_1 = 1/100 \times \mathbf{1}_2^{\text{T}}$, and $\alpha_1 = \mathbf{1}_2^{\text{T}}$;
—Scenario 2: $\mu_2 = (-4, -1, 2, 3)$, $\sigma_2 = 5 \times \mathbf{1}_5^{\text{T}}$, and $\alpha_2 = \mathbf{1}_4^{\text{T}}$;
—Scenario 3: $\mu_3 = (-10, 10, 20, 30, 40)$, $\sigma_3 = 50 \times \mathbf{1}_5^{\text{T}}$, and $\alpha_3 = \mathbf{1}_5^{\text{T}}$.

Table 2 Values of $\varDelta$ for scenarios 1, 2, and 3

γ	Scenario 1 $(\mu_1, \sigma_1, \alpha_1)$	Scenario 2 $(\mu_2, \sigma_2, \alpha_2)$	Scenario 3 $(\mu_3, \sigma_3, \alpha_3)$
4	1.3×10^{-4}	4.5×10^{-5}	3.3×10^{-5}
10	7.5×10^{-6}	2.4×10^{-6}	1.8×10^{-6}
15	2.1×10^{-6}	6.9×10^{-7}	5.0×10^{-7}
20	8.8×10^{-7}	2.8×10^{-7}	2.1×10^{-7}
50	5.4×10^{-8}	1.7×10^{-8}	1.3×10^{-8}
100	6.7×10^{-9}	2.1×10^{-9}	1.6×10^{-9}
500	5.3×10^{-11}	1.7×10^{-11}	1.2×10^{-11}

We may observe that the values of $\varDelta$ are all quite low — indicating a good approximation — and that the parameter γ is inversely related to $\varDelta$, and $\varDelta$ is quite unresponsive to changes in the values of μ_j, β_j, and α_j.

In Figure 1 we present examples of near-exact density and cumulative distribution functions for positive linear combinations of independent Gumbel random variables and in Figure 2 for sums of independent Gumbel random variables, all with the same scale parameter.

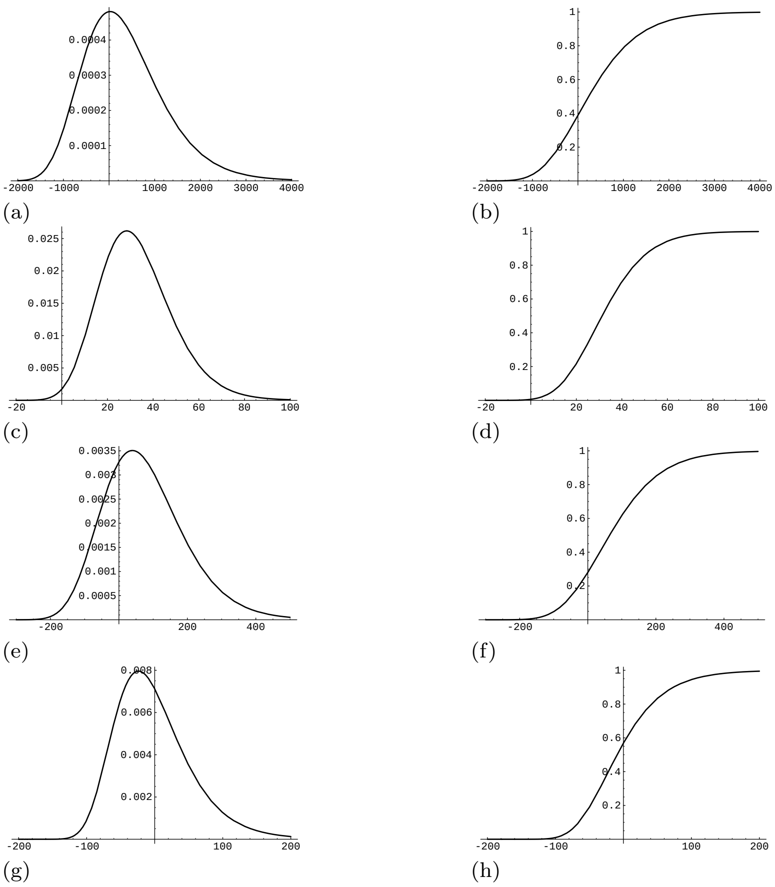

Fig. 1 Near-exact densities and distribution functions for the cases: (a–b) $\mu = (-20, -1, -50, 12, 40)$, $\sigma = (2, 1/2, 5/4, 10, 50)$, $\alpha = (2, 12, 24, 50, 10)$, and $\gamma = 6$; (c–d) $\mu = (2, 3, 4, 5^{1/2}, \pi, -6, -7, -7)$, $\sigma = (1/2, \pi, \exp\{1\}, 2^{1/2}, 1.2, 3.1, 2, 1)$, $\alpha = (1, 2, 3, 1/2, 5, 1, 1, 1, 1)$, and $\gamma = 8$; (e–f) $\mu = (1, 2, 3, -3, -2, -1)$, $\sigma = (1, 2, 3, 4, 5, 6)$, $\alpha = (2, 4, 6, 8, 10, 12)$, and $\gamma = 8$; (g–h) $\mu = (-2, -4)$, $\sigma = (5, 6)$, $\alpha = (3, 7)$, and $\gamma = 20$.

4 Conclusions

In this paper we have shown that the exact distribution of the linear combination of p independent Gumbel random variables is the same the distribution of the sum of two independent random variables, the first one corresponds to a linear combination of independent log Gamma distributions and the second one to a

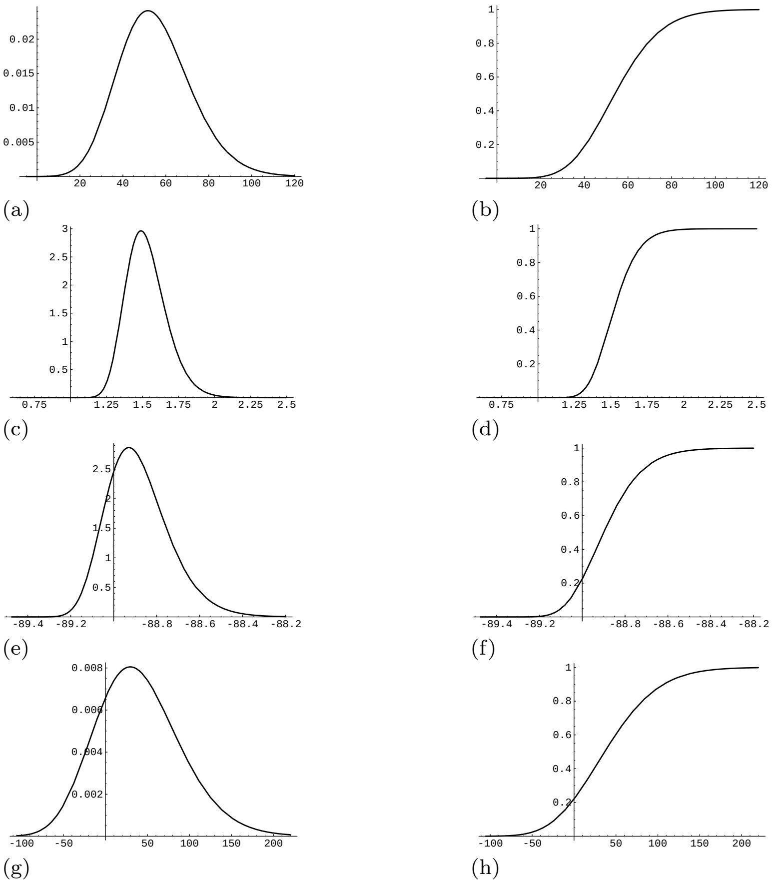

Fig. 2 Near-exact densities and distribution functions for the cases: (a–b) $\mu = (2,3,4,5,6,7,8)$, $\sigma = 5 \times \mathbf{1}_7^{\mathrm{T}}$, and $\gamma = 2$; (c–d) $\mu = (2/10, 3/10, 4/10, 5/10)$, $\sigma = 55/1000 \times \mathbf{1}_4^{\mathrm{T}}$, and $\gamma = 5$; (e–f) $\mu = (-29, -25, -35)$, $\sigma = 1/15 \times \mathbf{1}_3^{\mathrm{T}}$, and $\gamma = 7$; (g–h) $\mu = (-9, -5, -5, -7, 2, 3, 1/2)$, $\sigma = 15 \times \mathbf{1}_7^{\mathrm{T}}$, and $\gamma = 9$.

shifted Generalized Integer Gamma distribution. Using this result it was possible to derive a very accurate near-exact approximation which is at the same time very simple to use. We introduced the parameter γ in the representation of the distribution in study which works as a precision parameter since its value can be chosen based on the precision and speed desired for the calculations. Numerical studies presented show the hight quality of the near-exact distributions proposed for the linear combination of Gumbel random variables.

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A Appendix

Some Useful Results and Definitions on Distributions of Interest

We say that the random variable X has a Gamma distribution with shape parameter $r>0$ and rate parameter $\lambda>0$ if its probability density function is given by

$$f_X(x)=\frac{\lambda^r}{\Gamma(r)}\,\exp\{-\lambda x\}\,x^{r-1}\,,\qquad x>0$$

and we will denote this fact by $X\sim \text{Gamma}(r,\lambda)$.

Let $X_j \overset{\text{ind.}}{\sim} \text{Gamma}(r_j,\lambda_j)$ with shape parameters $r_j\in\mathbb{N}$ and rate parameters $\lambda_j\in\mathbb{R}^+$, all different, for $j=1,\ldots,p$. The Generalized Integer Gamma (GIG) distribution of depth $p\in\mathbb{N}$, introduced by Coelho (1998), is defined as the distribution of $Y=\sum_{j=1}^p X_j$, and we denote this by $Y\sim \text{GIG}(r,\lambda,p)$, for $r=(r_1,\ldots,r_p)$ and $\lambda=(\lambda_1,\ldots,\lambda_p)$. The density and distribution functions of Y are

$$f_Y(y;r,\lambda,p)=K\sum_{j=1}^{p}p_j(y)\exp\{-\lambda_j y\},\quad y>0,$$

$$F_Y(y;r,\lambda,p)=1-K\sum_{j=1}^{p}P_j(y)\exp\{-\lambda_j y\},\quad y>0,$$

where $K=\prod_{i=1}^p\lambda_i^{r_i}$,

$$p_i(y)=\sum_{k=1}^{r_i}c_{i,k}y^{k-1},\quad P_i(y)=\sum_{k=1}^{r_i}c_{i,k}(k-1)!\sum_{j=0}^{k-1}\frac{y^k}{j!\lambda_i^{k-j}}\,,$$

and the $c_{i,k}$ are given in (11)–(13) in Coelho (1998). The Generalized Near-Integer Gamma (GNIG) distribution of depth $(p+1)\in\mathbb{N}$, introduced by Coelho (2004), is defined has the distribution of $Y^\star=X^\star+\sum_{j=1}^p X_j$, where $X^\star$ is independent of $\sum_{j=1}^p X_j$, and $X^\star\sim\text{Gamma}(\rho,l)$, with $\rho\in\mathbb{R}^+\backslash\mathbb{N}$. We denote this by $Y^\star\sim\text{GNIG}(r^\star,\lambda^\star,p+1)$, where $r^\star=(r,\rho)$ and $\lambda^\star=(\lambda,l)$, and the corresponding density and distribution functions are

$$\begin{aligned}f_{Y^\star}(y;r^\star,\lambda^\star,p+1)&=Kl^\rho\sum_{j=1}^{p}\exp\{-\lambda_j y\}\\&\qquad\sum_{k=1}^{r_j}\left\{c_{j,k}\tfrac{\Gamma(k)}{\Gamma(k+\rho)}y^{k+\rho-1}\,{}_1F_1(\rho,k+\rho,-(l-\lambda_j)y)\right\},\\F_{Y^\star}(y;r^\star,\lambda^\star,p+1)&=\tfrac{l^\rho\,y^\rho}{\Gamma(\rho+1)}\,{}_1F_1(\rho,\rho+1,-ly)-Kl^\rho\sum_{j=1}^{p}\exp\{-\lambda_j y\}\\&\qquad\sum_{k=1}^{r_j}c^*_{j,k}\sum_{i=0}^{k-1}\tfrac{y^{r+i}\lambda_j^i}{\Gamma(\rho+1+i)}\,{}_1F_1(\rho,\rho+1+i,-(l-\lambda_j)y),\end{aligned}$$

for $y>0$ and where $c^*_{j,k}=(c_{j,k}\lambda_j^k)/\Gamma(k)$; in the above expressions ${}_1F_1(\cdot)$ denotes the Kummer confluent hypergeometric function.

The random variable $X^*=X+\theta$ is a shifted Gamma distribution with rate $\lambda\in\mathbb{R}^+$, shape $r\in\mathbb{R}^+$, and shift $\theta\in\mathbb{R}$, if $X\sim\text{Gamma}(r,\lambda)$, and we denote this by $X^*\sim\text{SGamma}(r,\lambda,\theta)$; the shifted GIG and GNIG distributions are analogously defined and denoted by $\text{SGIG}(r,\lambda,p,\theta)$ and $\text{SGNIG}(r^\star,\lambda^\star,p+1,\theta)$.

B Appendix

Computational implementation of the near-exact distribution developed

The computational implementation of the near-exact distribution proposed for the linear combination of Gumbel random variables may be made, for example, using the software Mathematica.

Fig. 3 Mathematica module for the cumulative distribution function of positive linear combinations of independent Gumbel random variables

```
LinearGumbelsCDF[alpha_,mu_,sigma_,gamma_,w_]:=Module[{l,rho,theta,mom,mom1,v,vs,n,isc,lambda,r,shift,c,g,P}
  mom=Table[SetPrecision[I^(-h)*D[Product[Gamma[gamma-I*t*sigma[[j]]*alpha[[j]]]/Gamma[gamma],
        {j,1,Length[mu]}],{t,h}]/.t->0,150],{h,1,3}];
  mom1=Table[SetPrecision[I^(-h)*D[l^rho*(l-I*t)^(-rho)*Exp[I*t*theta],{t,h}]/.t->0,150],{h,1,3}];
  {rho,l,theta}={rho,l,theta}/.Flatten[Solve[{mom[[1]]==mom1[[1]],mom[[2]]==mom1[[2]],
   mom[[3]]==mom1[[3]]},{rho,l,theta}]];
  v=Flatten[{Table[Table[((1+k)/(sigma[[j]]*alpha[[j]])),{k,0,gamma-2}],{j,1,Length[sigma]}]}];
  vs = Sort[v];n=Length[v];lambda={vs[[1]]};r={1};isc=1;
  Do[If[vs[[i]]==vs[[i-1]],{r[[isc]]=r[[isc]]+1},{isc=isc+1,lambda=Append[lambda,vs[[i]]],
    r=Append[r,1]}],{i, 2, n}];
  If[Count[r,_Integer]==Length[r] && And @@ Positive[r] && And @@ Positive[lambda], g = Length[r];
   shift = theta + Sum[alpha[[j]]*mu[[j]], {j, 1, Length[sigma]}];
   c=Table[Table[0,{j,1,Max[r]}],{i,1,g}];
   Table[c=ReplacePart[c,(Product[(lambda[[j]]-lambda[[i]])^(-r[[j]]),{j,1,i-1}]*
         Product[(lambda[[j]]-lambda[[i]])^(-r[[j]]),{j,i+1,g}])/(r[[i]]-1)!,{i,r[[i]]}],{i,1,g}];
   Table[Table[c=ReplacePart[c,Sum[((r[[i]]-k+j-1)!*(Sum[r[[h]]/(lambda[[i]]-lambda[[h]])^j,
    {h,1,i-1}]+Sum[r[[h]]/(lambda[[i]]-lambda[[h]])^j,{h,i+1,g}])*
    c[[i]][[r[[i]]-(k-j)]])/(r[[i]]-k-1)!,{j, 1, k}]/k,{i,r[[i]]-k}],{k,1,r[[i]]-1}],{i,1,g}];
   l^rho*(w - shift)^rho/Gamma[rho + 1]*Hypergeometric1F1[rho,rho+1,-l(w-shift)]
     -Product[lambda[[j]]^r[[j]],{j,1,g}]*l^rho*Sum[Exp[-lambda[[j]]*(w-shift)]*
     Sum[c[[j]][[k]]/lambda[[j]]^k*Gamma[k]*Sum[(w-shift)^(rho+i)*lambda[[j]]^i/Gamma[rho+1+i]
     *Hypergeometric1F1[rho,rho+1+i,-(l-lambda[[j]])*(w-shift)],{i,0,k-1}],{k,1,r[[j]]}],{j,1,g}]]]
```

Fig. 4 Mathematica module for the density function of positive linear combinations of independent Gumbel random variables

```
LinearGumbelsPDF[alpha_,mu_,sigma_,gamma_,w_]:=Module[{l,rho,theta,mom,mom1,v,vs,n,isc,lambda,r,shift,c,g,P}
  mom=Table[SetPrecision[I^(-h)*D[Product[Gamma[gamma-I*t*sigma[[j]]*alpha[[j]]]/Gamma[gamma],
        {j,1,Length[mu]}],{t,h}]/.t->0,150],{h,1,3}];
  mom1=Table[SetPrecision[I^(-h)*D[l^rho*(l-I*t)^(-rho)*Exp[I*t*theta],{t,h}]/.t->0,150],{h,1,3}];
  {rho,l,theta}={rho,l,theta}/.Flatten[Solve[{mom[[1]]==mom1[[1]],mom[[2]]==mom1[[2]],
   mom[[3]]==mom1[[3]]},{rho, l, theta}]];
  v=Flatten[{Table[Table[((1+k)/(sigma[[j]]*alpha[[j]])),{k,0,gamma-2}],{j,1,Length[sigma]}]}];
  vs=Sort[v];n=Length[v];lambda={vs[[1]]};r={1};isc=1;
  Do[If[vs[[i]]==vs[[i-1]],{r[[isc]]=r[[isc]]+1},{isc=isc+1,lambda=Append[lambda,vs[[i]]],
    r=Append[r,1]}],{i, 2, n}];
  If[Count[r,_Integer]==Length[r] && And @@ Positive[r] && And @@ Positive[lambda], g = Length[r];
   shift=theta+Sum[alpha[[j]]*mu[[j]],{j,1,Length[sigma]}];
   c=Table[Table[0,{j,1,Max[r]}],{i,1,g}];
   Table[c=ReplacePart[c,(Product[(lambda[[j]]-lambda[[i]])^(-r[[j]]),{j,1,i-1}]*
         Product[(lambda[[j]]-lambda[[i]])^(-r[[j]]),{j,i+1,g}])/(r[[i]]-1)!,{i,r[[i]]}],{i,1,g}];
   Table[Table[c=ReplacePart[c,Sum[((r[[i]]-k+j-1)!*(Sum[r[[h]]/(lambda[[i]]-lambda[[h]])^j,
    {h,1,i-1}]+Sum[r[[h]]/(lambda[[i]]-lambda[[h]])^j,{h,i+1,g}])*
    c[[i]][[r[[i]]-(k-j)]])/(r[[i]]-k-1)!,{j,1,k}]/k,{i,r[[i]]-k}],{k,1,r[[i]]-1}],{i,1,g}];
   Product[lambda[[j]]^r[[j]],{j,1,g}]*l^rho*Sum[Exp[-lambda[[j]]*(w-shift)]*Sum[c[[j]][[k]]*
   Gamma[k]/Gamma[k + rho]*(w-shift)^(k+rho-1)*Hypergeometric1F1[rho,k+rho,-(l-lambda[[j]])*
    (w-shift)],{k, 1, r[[j]]}], {j, 1, g}]]]
```

In Figures 3 and 4 we have Mathematica modules that may be used to compute respectively the cumulative distribution and density functions of W. This 'sample code' was developed with Mathematica 7.0. The code can be obtained from the first author (fjm@fct.unl.pt).

For example if we need to evaluate the cumulative distribution function of W at the value $w = 7.4165313$ for

$$\mu = (-1, 2, 3, 6) \quad ; \quad \sigma = (1/2, 5, 7/2, 3) \quad ; \quad \alpha = (1/2, 1/3, 1/4, 1/5) \quad ; \quad \gamma = 10$$

we should use

```
mu={-1,2,3,6};
sigma={1/2,5,7/2,3};
alpha={1/2,1/3,1/4,1/5};
gamma=10;
w=7.4165313;
SetPrecision[LinearGumbelsCDF[alpha,mu,sigma,gamma,w],20]
```

and the result is 0.89999999372369610112, which is obtained in more or less 0.3 seconds in a 2.00GHz processor. If we wish to plot the density and cumulative distribution function of W for

$$\mu = (-20, -1, -50, 12, 40) \quad ; \quad \sigma = (2, 1/2, 5/4, 10, 50) \quad ; \quad \alpha = (2, 12, 24, 50, 10) \quad ; \quad \gamma = 6;$$

we should use

```
mu={-20,-1,-50,12,40};
sigma={2,1/2,5/4,10,50};
alpha={2,12,24,50,10};
gamma= 6;
Plot[LinearGumbelsPDF[alpha, mu, sigma, gamma, w],{w,-2000,4000}]
Plot[LinearGumbelsCDF[alpha, mu, sigma, gamma, w],{w,-2000,4000}]
```

and the result should be the first two plots in Figure 1.

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