On the ranks of certain monoids of transformations that preserve a uniform partition

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Abstract

The purpose of this paper is to compute the ranks of the monoid $\mathcal{OR}_{m\times n}$ of all orientation-preserving or orientation-reversing full transformations on a chain with mn elements that preserve a uniform m-partition and of its submonoids $\mathcal{OP}_{m\times n}$ of all orientation-preserving transformations and $\mathcal{OD}_{m\times n}$ of all order-preserving or order-reversing full transformations. These three monoids are natural extensions of $\mathcal{O}_{m\times n}$, the monoid of all order-preserving full transformations on a chain with mn elements that preserve a uniform m-partition. Moreover, we also determine the ranks of certain semigroups of orientation-preserving full transformations with restricted ranges.

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Introduction and preliminaries

Let X be a set and denote by $\mathcal{T}(X)$ the monoid (under composition) of all full transformations on X. For $n \in \mathbb{N}$, let X_n be a chain with n elements, say $X_n = \{1 < 2 < \cdots < n\}$, and denote the monoid $\mathcal{T}(X_n)$ simply by $\mathcal{T}_n$. We say that a transformation α in $\mathcal{T}_n$ is *order-preserving* [*order-reversing*] if, for all $x, y \in X_n$, $x \le y$ implies $x\alpha \le y\alpha$ [$x\alpha \ge y\alpha$]. Notice that, the product of two order-preserving transformations or of two order-reversing transformations is order-preserving and the product of an order-preserving transformation by an order-reversing transformation is order-reversing. Denote by $\mathcal{O}_n$ the submonoid of $\mathcal{T}_n$ whose elements are order-preserving and by $\mathcal{OD}_n$ the submonoid of $\mathcal{T}_n$ whose elements are either order-preserving or order-reversing. Next, let $a = (a_1, a_2, \ldots, a_t)$ be a sequence of t ($t \ge 0$) elements from the chain X_n. We say that a is *cyclic* [*anti-cyclic*] if there exists no more than one index $i \in \{1, \ldots, t\}$ such that $a_i > a_{i+1}$ [$a_i < a_{i+1}$], where a_{t+1} denotes a_1. Let $\alpha \in \mathcal{T}_n$. We say that α is an *orientation-preserving* [*orientation-reversing*] transformation if the sequence of its images $(1\alpha, \ldots, n\alpha)$ is cyclic [anti-cyclic]. Like in the order case, the product of two orientation-preserving or of two orientation-reversing transformations is orientation-preserving and the product of an orientation-preserving transformation by an orientation-reversing transformation is orientation-reversing. Denote by $\mathcal{OP}_n$ the submonoid of $\mathcal{T}_n$ whose elements are orientation-preserving and by $\mathcal{OR}_n$ the submonoid of $\mathcal{T}_n$ whose elements are either orientation-preserving or orientation-reversing.

Semigroups of order-preserving transformations have long been considered in the literature. In 1962, Aĭzenštat [2] gave a presentation for $\mathcal{O}_n$, from which it can be deduced that $\mathcal{O}_n$ has only one non-trivial automorphism, for $n > 1$. Also in 1962, Aĭzenštat [1] showed that the non-trivial congruences of $\mathcal{O}_n$ are exactly the Rees congruences. Some years later, in 1971, Howie [16] studied some combinatorial and algebraic properties of $\mathcal{O}_n$, in particular, he showed that $\mathcal{O}_n$ is generated by idempotents of defect one and has F_{2n} idempotents, where F_{2n} is the $2n^{\text{th}}$ Fibonacci number. Later, in 1992, Gomes and Howie [15] revisited the semigroup $\mathcal{O}_n$ and computed its rank and idempotent rank (which are n and $2n-2$, respectively). Recall that the [idempotent] rank of a finite [idempotent generated] monoid is the cardinality of a least-size [idempotent] generating set. More recently, Fernandes et al. [11] characterized the endomorphisms of $\mathcal{O}_n$. The notion of an orientation-preserving transformation was introduced by McAlister in [20] and, independently, by Catarino and Higgins in [6]. Several properties of the monoids $\mathcal{OP}_n$ and $\mathcal{OR}_n$ have been investigated in these two papers. A presentation for the monoid $\mathcal{OP}_n$, in terms of $2n-1$ generators, was given by Catarino in [5]. A new presentation for $\mathcal{OP}_n$, in terms of 2 (its rank) generators, was found by Arthur and Ruškuc [4], who also exhibited a presentation for the monoid $\mathcal{OR}_n$, in terms of 3 (its rank) generators. Finally, regarding the monoid $\mathcal{OD}_n$, a presentation was given by Fernandes et al. in [9]. Its rank, computed in [10] by the same authors, is $\lceil n/2 \rceil + 1$.

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Now, let ρ be an equivalence relation on a set X and denote by $\mathcal{T}_\rho(X)$ the submonoid of $\mathcal{T}(X)$ of all transformations that preserve the equivalence relation ρ, i.e. $\mathcal{T}_\rho(X) = \{\alpha \in \mathcal{T}(X) \mid (a\alpha, b\alpha) \in \rho, \text{for all } (a,b) \in \rho\}$. This monoid was studied by Huisheng in [18] who determined its regular elements and described its Green's relations.

Let $m, n \in \mathbb{N}$. Of particular interest is the submonoid $\mathcal{T}_{m\times n} = \mathcal{T}_\rho(X_{mn})$ of $\mathcal{T}_{mn}$, with ρ the equivalence relation on X_{mn} defined by $\rho = (A_1 \times A_1) \cup (A_2 \times A_2) \cup \cdots \cup (A_m \times A_m)$, where $A_i = \{(i-1)n+1, \ldots, in\}$, for $i \in \{1, \ldots, m\}$. Notice that the ρ-classes A_i, with $1 \leq i \leq m$, form a uniform m-partition of X_{mn}.

Regarding the rank of $\mathcal{T}_{m\times n}$, first, Huisheng [17] proved that it is at most 6 and, later, Araújo and Schneider [3] improved this result by showing that, for $|X_{mn}| \geq 3$, the rank of $\mathcal{T}_{m\times n}$ is precisely 4. The ranks of its partial and (partial) injective counterparts were determined by the first author together with Cicalò and Schneider [7].

Finally, denote by $\mathcal{OR}_{m\times n}$ the submonoid of $\mathcal{T}_{m\times n}$ of all orientation-preserving or orientation-reversing transformations, i.e. $\mathcal{OR}_{m\times n} = \mathcal{T}_{m\times n} \cap \mathcal{OR}_{mn}$. Similarly, let $\mathcal{OD}_{m\times n} = \mathcal{T}_{m\times n} \cap \mathcal{OD}_{mn}$, $\mathcal{OP}_{m\times n} = \mathcal{T}_{m\times n} \cap \mathcal{OP}_{mn}$ and $\mathcal{O}_{m\times n} = \mathcal{T}_{m\times n} \cap \mathcal{O}_{mn}$.

Example Consider the following transformations of $\mathcal{T}_{12}$:

$$\alpha_1 = \left(\begin{array}{cccc|cccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 9 & 11 & 10 & 12 & 1 & 3 & 3 & 2 & 5 & 5 & 7 & 8 \end{array}\right); \quad \alpha_2 = \left(\begin{array}{cccc|cccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 8 & 8 & 8 & 6 & 6 & 5 & 5 & 5 & 12 & 12 & 11 & 10 \end{array}\right);$$

$$\alpha_3 = \left(\begin{array}{cccc|cccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 11 & 11 & 10 & 10 & 10 & 9 & 9 & 9 & 4 & 3 & 3 & 1 \end{array}\right); \quad \alpha_4 = \left(\begin{array}{cccc|cccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 7 & 7 & 7 & 8 & 8 & 8 & 5 & 5 & 5 & 6 & 6 & 7 \end{array}\right);$$

$$\alpha_5 = \left(\begin{array}{cccc|cccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 1 & 1 & 2 & 3 & 3 & 4 & 4 & 10 & 11 & 11 & 11 \end{array}\right); \quad \alpha_6 = \left(\begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 1 & 2 & 3 & 5 & 5 & 6 & 9 & 9 & 10 & 10 & 11 \end{array}\right).$$

Then, we have: $\alpha_1 \in \mathcal{T}_{3\times 4}$, but $\alpha_1 \notin \mathcal{OR}_{3\times 4}$; $\alpha_2 \in \mathcal{OR}_{3\times 4}$, but $\alpha_2 \notin \mathcal{OP}_{3\times 4}$; $\alpha_3 \in \mathcal{OD}_{3\times 4}$, but $\alpha_3 \notin \mathcal{O}_{3\times 4}$; $\alpha_4 \in \mathcal{OP}_{3\times 4}$, but $\alpha_4 \notin \mathcal{O}_{3\times 4}$; $\alpha_5 \in \mathcal{O}_{3\times 4}$; and, finally, $\alpha_6 \notin \mathcal{T}_{3\times 4}$.

In [19] Huisheng and Dingyu described the regular elements and the Green relations of $\mathcal{O}_{m\times n}$. On the other hand, in [12] the authors proved that the monoid $\mathcal{O}_{m\times n}$ has rank $2mn - n$. A description of the regular elements and a characterization of the Green relations of the monoid $\mathcal{OP}_{m\times n}$ were given by Sun et al. in [21]. The cardinals of the monoids $\mathcal{OR}_{m\times n}$, $\mathcal{OP}_{m\times n}$, $\mathcal{OD}_{m\times n}$ and $\mathcal{O}_{m\times n}$ were determined by the authors in [13].

In this paper, we continue the work of [12] and compute the ranks of the monoids $\mathcal{OP}_{m\times n}$, $\mathcal{OD}_{m\times n}$ and $\mathcal{OR}_{m\times n}$ (Sections 2, 3 and 4, respectively). In order to help achieving this goal, we use the wreath product description of $\mathcal{T}_{m\times n}$, due to Araújo and Schneider [3], that we recall in the beginning of Section 2. On the other hand, since it will be useful to determine the rank of $\mathcal{OP}_{m\times n}$, in Section 1, we find generating sets (and the ranks) of certain subsemigroups of $\mathcal{OP}_n$ with restricted ranges.

1 On the semigroups $\mathcal{OP}_{n,r}$

Let $n \in \mathbb{N}$ and $1 \leq r \leq n$. Consider the subsemigroup with restricted range $\mathcal{OP}_{n,r} = \{\alpha \in \mathcal{OP}_n \mid \mathrm{Im}(\alpha) \subseteq \{1, \ldots, r\}\}$ of $\mathcal{OP}_n$. Recall that the ranks and other properties of the subsemigroups of restricted range of $\mathcal{PT}_n$, $\mathcal{T}_n$ and $\mathcal{I}_n$ were studied by Fernandes and Sanwong in [14]. In this section, we determine a set of generators of $\mathcal{OP}_{n,r}$ that we will use in the next section. Moreover, we deduce that $\mathcal{OP}_{n,r}$ has rank equal to $\binom{n}{r}$, for $2 \leq r \leq n-1$.

Notice that, $\mathcal{OP}_{n,1}$ is a trivial semigroup and $\mathcal{OP}_{n,n} = \mathcal{OP}_n$. Therefore, in what follows, we consider $2 \leq r \leq n-1$.

We begin by showing that $\mathcal{OP}_{n,r}$ is generated by its elements of rank r.

Lemma 1.1 *For $1 \leq k < r$, any transformation of $\mathcal{OP}_{n,r}$ of rank k is a product of elements of $\mathcal{OP}_{n,r}$ of rank $k+1$.*

Proof. Let $\alpha = \left(\begin{array}{c|c|c|c} I_1 & I_2 & \cdots & I_k \\ a_1 & a_2 & \cdots & a_k \end{array}\right)$ be an element of $\mathcal{OP}_{n,r}$ of rank k, where $I_1, I_2, \ldots, I_k$ are the kernel classes of α in order $1 \in I_1$ and $\min I_i < \min I_{i+1}$, for $i = 1, \ldots, k-1$. Notice that $I_2, \ldots, I_k$ are intervals and I_1 is an interval if and only if $n \in I_k$ (otherwise I_1 is a union of two intervals). Observe also that $(a_1, a_2, \ldots, a_k)$ is a k-cycle. On the other hand, as $k < n$, there exits $j \in \{1, \ldots, k\}$ such that $|I_j| > 1$.

Consider $\gamma = \left(\begin{array}{ccc|ccc|ccc} 1 & \cdots & k-j & k-j+1 & \cdots & k & k+1 & \cdots & n \\ a_{j+1} & \cdots & a_k & a_1 & \cdots & a_j & a_j & \cdots & a_j \end{array}\right)$. Clearly, $\gamma \in \mathcal{OP}_{n,r}$ and γ has rank k. Next, if $2 \leq j \leq k$, let $\beta = \left(\begin{array}{c|c|c|c|c|c|c|c} I_1 & \cdots & I_{j-1} & \min I_j & I_j \backslash \{\min I_j\} & I_{j+1} & \cdots & I_k \\ k-j+1 & \cdots & k-1 & k & k+1 & 1 & \cdots & k-j \end{array}\right)$; if $j = 1$ and $n \in I_k$, let $\beta = \left(\begin{array}{c|ccc|c|c|c} 1 & 2 & \cdots & \max I_1 & I_2 & \cdots & I_k \\ k & k+1 & \cdots & k+1 & 1 & \cdots & k-1 \end{array}\right)$; and, if $j = 1$ and $n \in I_1$, let $\beta = \left(\begin{array}{c|c|c|c|c} I_1' & I_2 & \cdots & I_k & I_1'' \\ k+1 & 1 & \cdots & k-1 & k \end{array}\right)$, where I_1' and I_1'' are intervals such that $I_1' \cup I_1'' = I_1$ and $\max I_1' < \min I_1''$ (notice that, we also have $\max I_k < \min I_1''$). Hence, in all cases, it is a routine matter to check that β is an element of $\mathcal{OP}_{n,r}$ of rank $k+1$ and $\alpha = \beta\gamma$.

Now, we focus our attention on γ. Let $(b_1, \ldots, b_k)$ be the k-cycle $(a_{j+1}, \ldots, a_k, a_1, \ldots, a_j)$. Observe that, with this notation, we have $\gamma = \begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & n \\ b_1 & \cdots & b_k & b_k & \cdots & b_k \end{pmatrix}$. Take $b \in \{1, \ldots, r\} \setminus \mathrm{Im}(\gamma)$. If $b_k < b < b_1$ or $b_1 < b_k < b$, let $\gamma_1 = \begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & n \\ 1 & \cdots & k & k+1 & \cdots & k+1 \end{pmatrix}$ and $\gamma_2 = \begin{pmatrix} 1 & \cdots & k & k+1 & k+2 & \cdots & n \\ b_1 & \cdots & b_k & b_k & b & \cdots & b \end{pmatrix}$. On the other hand, if $b_i < b < b_{i+1}$ or $b < b_{i+1} < b_i$ or $b_{i+1} < b_i < b$, for some $i \in \{1, \ldots, k-1\}$, let

$$\gamma_1 = \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & k & k+1 & \cdots & n \\ k-i+2 & \cdots & k & k+1 & 1 & \cdots & k-i & k-i+1 & \cdots & k-i+1 \end{pmatrix}$$

and

$$\gamma_2 = \begin{pmatrix} 1 & \cdots & k-i & k-i+1 & k-i+2 & \cdots & k+1 & k+2 & \cdots & n \\ b_{i+1} & \cdots & b_k & b_k & b_1 & \cdots & b_i & b & \cdots & b \end{pmatrix}$$

(notice that $k < n-1$, whence $k+2 \leq n$). Then, in both cases, it is easy to show that $\gamma_1, \gamma_2 \in \mathcal{OP}_{n,r}$, γ_1 and γ_2 have rank $k+1$ and $\gamma = \gamma_1\gamma_2$.

Therefore, we proved that $\alpha = \beta\gamma_1\gamma_2$, with β, γ_1 and γ_2 elements of $\mathcal{OP}_{n,r}$ of rank $k+1$, as required. $\square$

From this lemma, by induction on the rank of the transformations, we may deduce that $\mathcal{OP}_{n,r}$ is generated by its elements of rank r, as announced above.

Next, let $g_{n,r} = \left(\begin{array}{ccccc|ccc} 1 & 2 & \cdots & r-1 & r & r+1 & \cdots & n \\ 2 & 3 & \cdots & r & 1 & 1 & \cdots & 1 \end{array}\right) \in \mathcal{OP}_{n,r}$. Hence, we have:

Lemma 1.2 *Let α and β be two elements of $\mathcal{OP}_{n,r}$ of rank r such that* $\mathrm{Ker}(\beta) = \mathrm{Ker}(\alpha)$. *Then $\beta = \alpha g_{n,r}^k$, for some $k \in \{0, \ldots, r-1\}$.*

Proof. Take $\alpha = \left(\begin{array}{c|c|c|c} I_1 & I_2 & \cdots & I_r \\ a_1 & a_2 & \cdots & a_r \end{array}\right)$ and $\beta = \left(\begin{array}{c|c|c|c} I_1 & I_2 & \cdots & I_r \\ b_1 & b_2 & \cdots & b_r \end{array}\right)$, where $I_1, I_2, \ldots, I_r$ are the kernel classes of α and β in order $1 \in I_1$ and $\min I_i < \min I_{i+1}$, for $i = 1, \ldots, r-1$. Then, as $(a_1, a_2, \ldots, a_r)$ and $(b_1, b_2, \ldots, b_r)$ are two r-cycles of $\{1, \ldots, r\}$, we have $(a_1, \ldots, a_r) = (i+1, \ldots, r, 1, \ldots, i)$ and $(b_1, \ldots, b_r) = (j+1, \ldots, r, 1, \ldots, j)$, for some $1 \leq i, j \leq r$. Take $k = j - i$, if $i \leq j$, and $k = r - i + j$, otherwise. Hence, $k \in \{0, \ldots, r-1\}$ and it is a routine matter to prove that $\beta = \alpha g_{n,r}^k$, as required. $\square$

Now, notice that, if α is an element of $\mathcal{OP}_{n,r}$ of rank r and α_1 and α_2 are two elements of $\mathcal{OP}_{n,r}$ such that $\alpha = \alpha_1\alpha_2$, then $\mathrm{Ker}(\alpha_1) = \mathrm{Ker}(\alpha)$. On the other hand, it is clear that the number of distinct kernels of transformations of $\mathcal{OP}_{n,r}$ of rank r coincides with the number of distinct kernels of transformations of $\mathcal{OP}_n$ of rank r, which is precisely $\binom{n}{r}$ (see [6]). These observations, together with the previous two lemmas, prove the following result.

Theorem 1.3 *For $2 \leq r \leq n-1$, the semigroup $\mathcal{OP}_{n,r}$ is generated by any subset of transformations of rank r containing at least one element from each distinct kernel. Furthermore, $\mathcal{OP}_{n,r}$ has rank equal to $\binom{n}{r}$.*

2 The rank of the monoid $\mathcal{OP}_{m\times n}$

Let $m, n \geq 2$. Following [3], we define the wreath product $\mathcal{T}_n \wr \mathcal{T}_m$ of $\mathcal{T}_n$ and $\mathcal{T}_m$ as being the monoid with underlying set $\mathcal{T}_n^m \times \mathcal{T}_m$ and multiplication defined by

$$(\alpha_1, \ldots, \alpha_m; \beta)(\alpha_1', \ldots, \alpha_m'; \beta') = (\alpha_1\alpha_{1\beta}', \ldots, \alpha_m\alpha_{m\beta}'; \beta\beta'),$$

for all $(\alpha_1, \ldots, \alpha_m; \beta)$, $(\alpha_1', \ldots, \alpha_m'; \beta') \in \mathcal{T}_n^m \times \mathcal{T}_m$.

Let $\alpha \in \mathcal{T}_{m\times n}$ and let $\beta \in \mathcal{T}_m$ be the *quotient* map of α by ρ, i.e. for all $j \in \{1, \ldots, m\}$, we have $A_j\alpha \subseteq A_{j\beta}$. For each $j \in \{1, \ldots, m\}$, define $\alpha_j \in \mathcal{T}_n$ by

$$k\alpha_j = ((j-1)n + k)\alpha - (j\beta - 1)n, \qquad (1)$$

for all $k \in \{1, \ldots, n\}$. Let $\overline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m; \beta) \in \mathcal{T}_n^m \times \mathcal{T}_m$. With this notation, the function

$$\begin{array}{rccc} \psi: & \mathcal{T}_{m\times n} & \longrightarrow & \mathcal{T}_n \wr \mathcal{T}_m \\ & \alpha & \longmapsto & \overline{\alpha} \end{array}$$

is an isomorphism (see [3, Lemma 2.1]).

Notice that, from (1), we have $k\alpha_j < \ell\alpha_j$ if and only if $((j-1)n+k)\alpha < ((j-1)n+\ell)\alpha$, for all $1 \le k, \ell \le n$ and $j \in \{1, \dots, m\}$. Furthermore, if $j\beta = (j+1)\beta$, for some $j \in \{1, \dots, m-1\}$, then $n\alpha_j < 1\alpha_{j+1}$ if and only if $(jn)\alpha < (jn+1)\alpha$. Also, if $m\beta = 1\beta$, then $n\alpha_m < 1\alpha_1$ if and only if $(mn)\alpha < 1\alpha$.

Now, admit that α is an orientation-preserving transformation. Then,

1. $1\alpha \le \cdots \le (mn)\alpha$; or

2. $(r+1)\alpha \le \cdots \le (mn)\alpha \le 1\alpha \le \cdots \le r\alpha$ and $r\alpha > (r+1)\alpha$, for some $r \in \{1, \dots, mn-1\}$.

In the first case (notice that α is order-preserving), clearly, $\alpha_j \in \mathcal{O}_n$, for all $j \in \{1, \dots, m\}$. Next, suppose that α satisfies the second condition. If $r \in A_j \backslash \{jn\}$, for some $j \in \{1, \dots, m\}$, then $\alpha_j \in \mathcal{OP}_n \backslash \mathcal{O}_n$ and $\alpha_i \in \mathcal{O}_n$, for all $i \in \{1, \dots, m\} \backslash \{j\}$. Furthermore, $\mathrm{Im}(\alpha) \subseteq A_j\alpha$, whence β is constant. Otherwise (i.e. $r = jn$, for some $j \in \{1, \dots, m-1\}$), it is clear that we have $\alpha_i \in \mathcal{O}_n$, for all $i \in \{1, \dots, m\}$.

On the other hand, also as a consequence of (1), if $(in)\alpha \le (jn)\alpha$ then $i\beta \le j\beta$, for all $1 \le i, j \le m$. In fact, suppose that $i\beta > j\beta$, for some $1 \le i, j \le m$. Then, $i\beta = j\beta + t$, for some $t \ge 1$, and so $(i\beta)n = (j\beta)n + tn$. Hence $(in)\alpha = ((i-1)n+n)\alpha = n\alpha_i + (i\beta-1)n = n\alpha_i + (j\beta-1)n + tn > n\alpha_j + (j\beta-1)n = ((j-1)n+n)\alpha = (jn)\alpha$, as required. Now, if α is orientation-preserving then, as any subsequence of a cyclic sequence is also cyclic (see [8, Proposition 2.1]), the sequence $(n\alpha, (2n)\alpha, \dots, (mn)\alpha)$ is cyclic and so, by the above observation, the sequence $(1\beta, 2\beta, \dots, m\beta)$ is also cyclic, i.e. $\beta \in \mathcal{OP}_m$.

Recall that the authors showed in [12, Lemma 1.2] that

$$\mathcal{O}_{m\times n}\psi = \{(\alpha_1, \dots, \alpha_m; \beta) \in \mathcal{O}_n^m \times \mathcal{O}_m \mid j\beta = (j+1)\beta \text{ implies } n\alpha_j \le 1\alpha_{j+1}, \text{ for all } j \in \{1, \dots, m-1\}\}. \quad (2)$$

Considering addition modulo m (in particular, $m+1 = 1$), for $\mathcal{OP}_{m\times n}$, we have:

Proposition 2.1 *A $(m+1)$-tuple $(\alpha_1, \alpha_2, \dots, \alpha_m; \beta)$ of $\mathcal{T}_n^m \times \mathcal{T}_m$ belongs to $\mathcal{OP}_{m\times n}\psi$ if and only if it satisfies one of the following conditions:*

1. *β is a non-constant transformation of $\mathcal{OP}_m$,*
   for all $i \in \{1, \dots, m\}$, $\alpha_i \in \mathcal{O}_n$ and,
   for all $j \in \{1, \dots, m\}$, $j\beta = (j+1)\beta$ implies $n\alpha_j \le 1\alpha_{j+1}$;

2. *β is a constant transformation,*
   for all $i \in \{1, \dots, m\}$, $\alpha_i \in \mathcal{O}_n$ and
   there exists at most one index $j \in \{1, \dots, m\}$ such that $n\alpha_j > 1\alpha_{j+1}$;

3. *β is a constant transformation,*
   there exists one index $i \in \{1, \dots, m\}$ such that $\alpha_i \in \mathcal{OP}_n \setminus \mathcal{O}_n$ and, for all $j \in \{1, \dots, m\} \setminus \{i\}$, $\alpha_j \in \mathcal{O}_n$ and,
   for all $j \in \{1, \dots, m\}$, $n\alpha_j \le 1\alpha_{j+1}$.

Proof. We will take into consideration, several times, the observations stated above.

First, assuming that a $(m+1)$-tuple $(\alpha_1, \dots, \alpha_m; \beta)$ satisfies 1, 2 or 3, it is just a routine matter to check that, if $\alpha \in \mathcal{T}_{m\times n}$ is such that $\alpha\psi = (\alpha_1, \dots, \alpha_m; \beta)$ then $\alpha \in \mathcal{OP}_{m\times n}$.

Conversely, let $\alpha \in \mathcal{OP}_{m\times n}$ and take $\overline{\alpha} = \alpha\psi = (\alpha_1, \dots, \alpha_m; \beta)$.

If α is order-preserving then, by (2), $(\alpha_1, \dots, \alpha_m; \beta) \in \mathcal{O}_n^m \times \mathcal{O}_m$ and, for all $j \in \{1, \dots, m-1\}$, $j\beta = (j+1)\beta$ implies $n\alpha_j \le 1\alpha_{j+1}$. If β is not constant, then $m\beta \ne 1\beta$ and so the $(m+1)$-tuple $\overline{\alpha}$ satisfies 1. Otherwise, $\overline{\alpha}$ satisfies 2.

Next, suppose that $(r+1)\alpha \le \cdots \le (mn)\alpha \le 1\alpha \le \cdots \le r\alpha$ and $r\alpha > (r+1)\alpha$, for some $r \in \{1, \dots, mn-1\}$. If $r \in A_j \setminus \{jn\}$, for some $j \in \{1, \dots, m\}$, it is easy to deduce that $\overline{\alpha}$ satisfies 3. On the other hand, if $r = jn$, for some $j \in \{1, \dots, m-1\}$, it is easy to show that $\overline{\alpha}$ satisfies 1, if β is not constant, and that $\overline{\alpha}$ satisfies 2, otherwise. $\square$

Let $\alpha \in \mathcal{OP}_{m\times n}$. For $i \in \{1, 2, 3\}$, we say that α and $\alpha\psi$ are of *type* i if $\alpha\psi$ satisfies the condition i. of the previous proposition. Notice that, if $(\alpha_1, \dots, \alpha_m; \beta) = \alpha\psi$ is of type 2 and, for all $j \in \{1, \dots, m\}$, $n\alpha_j \le 1\alpha_{j+1}$, then α must be a constant transformation.

Moreover, as clearly the product of $(m+1)$-tuples of types 1 or 2 (respectively, 2 or 3) cannot be a $(m+1)$-tuple of type 3 (respectively, 1), then the subset $\overline{M}$ (respectively, $\overline{N}$) of $\mathcal{OP}_{m\times n}\psi$ of all $(m+1)$-tuples of types 1 or 2 (respectively, 2 or 3) is a submonoid (respectively, subsemigroup) of $\mathcal{OP}_{m\times n}\psi$.

Let $M = \overline{M}\psi^{-1}$. Hence, clearly, M is the submonoid of $\mathcal{OP}_{m\times n}$ whose elements are the order-preserving transformations (and so, in particular, M contains $\mathcal{O}_{m\times n}$) and the transformations $\alpha \in \mathcal{OP}_{m\times n}$ such that $(jn+1)\alpha \le \cdots \le (mn)\alpha \le 1\alpha \le \cdots \le (jn)\alpha$ and $(jn)\alpha > (jn+1)\alpha$, for some $j \in \{1,\ldots,m-1\}$.

Recall that, being g_n the n-cycle $\left(\begin{array}{ccccc} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{array}\right) \in \mathcal{OP}_n$, each element $s \in \mathcal{OP}_n$ admits a factorization $s = g_n^j u$, with $0 \le j \le n-1$ and $u \in \mathcal{O}_n$, which is unique unless s is constant (see [6]).

Now, consider the permutations (of $\{1,\ldots,mn\}$)

$$g = g_{mn} = \left(\begin{array}{ccccc} 1 & 2 & \cdots & mn-1 & mn \\ 2 & 3 & \cdots & mn & 1 \end{array}\right) \in \mathcal{OP}_{mn}$$

and

$$f = g^n = \left(\begin{array}{ccc|ccc|ccc} 1 & \cdots & n & n+1 & \cdots & mn-n & mn-n+1 & \cdots & mn \\ n+1 & \cdots & 2n & 2n+1 & \cdots & mn & 1 & \cdots & n \end{array}\right) \in \mathcal{OP}_{m\times n}.$$

Let $\alpha \in M \setminus \mathcal{O}_{m\times n}$ and take $j \in \{1,\ldots,m-1\}$ such that $(jn)\alpha > (jn+1)\alpha$. Then, as $(jn+1)\alpha \le \cdots \le (mn)\alpha \le 1\alpha \le \cdots \le (jn)\alpha$, it is clear that $f^j\alpha \in \mathcal{O}_{m\times n}$. Thus, we have:

Lemma 2.2 *Each element $\alpha \in M$ admits a factorization $\alpha = f^j\gamma$, with $0 \le j \le m-1$ and $\gamma \in \mathcal{O}_{m\times n}$, which is unique unless α is constant. In particular, the monoid M is generated by $\mathcal{O}_{m\times n}$ and f.*

Notice that, the uniqueness stated in the previous lemma follows immediately from the fact that f is a power of g and from Catarino and Higgins's result mentioned above.

Now, let $N = \overline{N}\psi^{-1}$. Clearly, N is the subsemigroup of $\mathcal{OP}_{m\times n}$ whose elements are the transformations $\alpha \in \mathcal{OP}_{m\times n}$ such that $\mathrm{Im}(\alpha) \subseteq A_j$, for some $j \in \{1,\ldots,m\}$. Next, we justify the study made in the previous section by considering $\mathcal{OP}_{mn,n}$, which is a subsemigroup of N. For $j \in \{1,\ldots,m\}$, let $\overline{\nu}_j = (1,\gamma_2,\ldots,\gamma_m;\beta_j)$, where $\gamma_2 = \cdots = \gamma_m = \left(\begin{array}{ccc} 1 & \cdots & n \\ n & \cdots & n \end{array}\right)$ and $\beta_j = \left(\begin{array}{ccc} 1 & \cdots & m \\ j & \cdots & j \end{array}\right)$. Clearly, $\overline{\nu}_j \in \overline{N}$, for all $j \in \{1,\ldots,m\}$. Next, let $\overline{\alpha} = (\alpha_1,\ldots,\alpha_m;\beta_j) \in \overline{N}$, with $j \in \{1,\ldots,m\}$. Then $\overline{\gamma} = (\alpha_1,\ldots,\alpha_m;\beta_1) \in \mathcal{OP}_{mn,n}\psi$ and $\overline{\alpha} = \overline{\gamma}\,\overline{\nu}_j$. On the other hand, noticing that $f\psi = (1,\ldots,1;g_m)$, we also have $\overline{\alpha}(f\psi)^{m-j+1} = \overline{\gamma}$, i.e. $\overline{\alpha} = \overline{\gamma}(f\psi)^{j-1}$.

Thus, being ν_j the element of N such that $\nu_j\psi = \overline{\nu}_j$, with $j \in \{1,\ldots,m\}$, we have:

Lemma 2.3 *The semigroup N is generated by $\mathcal{OP}_{mn,n} \cup \{\nu_2,\ldots,\nu_m\}$. Moreover, every element of N is a product of an element of $\mathcal{OP}_{mn,n}$ by a power of f.*

Next, for $j \in \{1,\ldots,n-1\}$, let

$$p_j = \left(\begin{array}{ccccccc|cc} 1 & 2 & \cdots & n-j & n-j+1 & \cdots & n & n+1 & \cdots \\ j+1 & j+2 & \cdots & n & 1 & \cdots & 1 & 1 & \cdots \end{array}\right.$$
$$\left.\begin{array}{cc|cccccc} \cdots & (m-1)n & (m-1)n+1 & \cdots & mn-j & mn-j+1 & \cdots & mn \\ \cdots & 1 & 1 & \cdots & 1 & 2 & \cdots & j+1 \end{array}\right) \in \mathcal{OP}_{mn,n}.$$

Notice that

$$p_1^i = \left(\begin{array}{cccccc|ccc|cccc} 1 & \cdots & n-i & n-i+1 & \cdots & n & n+1 & \cdots & (m-1)n & (m-1)n+1 & \cdots & mn-1 & mn \\ i+1 & \cdots & n & 1 & \cdots & i & i & \cdots & i & i & \cdots & i & i+1 \end{array}\right),$$

for $i \in \{1,\ldots,n-1\}$, and

$$p_1^n = \left(\begin{array}{ccc|ccc|cccc} 1 & \cdots & n & n+1 & \cdots & (m-1)n & (m-1)n+1 & \cdots & mn-1 & mn \\ 1 & \cdots & n & n & \cdots & n & n & \cdots & n & 1 \end{array}\right),$$

is a right identity of $\mathcal{OP}_{mn,n}$.

Lemma 2.4 *Any transformation of $\mathcal{OP}_{mn,n}$ is a product of elements of $M \cup \{p_j \mid 1 \le j \le \lceil \frac{n-1}{2} \rceil\}$.*

Proof. By Theorem 1.3, it suffices to consider only transformations of $\mathcal{OP}_{mn,n}$ with rank n. Let γ be such a transformation.

STEP 1. Let $i = 1\gamma$ and $\alpha = \gamma p_1^{n-i+1}$. Then, $1\alpha = 1\gamma p_1^{n-i+1} = ip_1^{n-i+1} = 1$ and $\gamma = \alpha p_1^{n+i-1}$. If $\alpha \in M$ then γ satisfies the statement of the lemma.

Therefore, suppose that $\alpha \notin M$. Hence, $(mn)\alpha = 1$ (otherwise $(mn)\alpha = n$, whence $\alpha \in \mathcal{O}_{mn}$ and so $\alpha \in M$). Let $r \in \{1, \dots, mn\}$ be the least integer such that $\{r, \dots, mn\}\alpha = \{1\}$. As α also has rank n and $1\alpha = 1$, then $r \geq n+1$. Thus, $r = (t-1)n + k + 1$, for some $t \in \{2, \dots, m\}$ and $k \in \{1, \dots, n-1\}$ (notice that, if $k = 0$ then $\alpha \in M$).

Let $j = ((t-1)n)\alpha - 1$ (notice that $0 \leq j \leq n-1$). If $j = 0$ then

$$\alpha = \left(\begin{array}{ccc|cccc|ccc} 1 & \cdots & (t-1)n & (t-1)n+1 & \cdots & tn-1 & tn & tn+1 & \cdots & mn \\ 1 & \cdots & 1 & 2 & \cdots & n & 1 & 1 & \cdots & 1 \end{array}\right),$$

whence

$$\alpha p_1^{n-1} = \left(\begin{array}{ccc|cccc|ccc} 1 & \cdots & (t-1)n & (t-1)n+1 & \cdots & tn-1 & tn & tn+1 & \cdots & mn \\ n & \cdots & n & 1 & \cdots & n-1 & n & n & \cdots & n \end{array}\right) \in M$$

and so, as $\gamma = \alpha p_1^{n+i-1} = (\alpha p_1^{n-1})p_1^i$, in this case γ also satisfies the statement of the lemma. Otherwise, let $\beta \in \mathcal{T}_{mn}$ be defined by

$$x\beta = \begin{cases} mn - (j+1-x\alpha) & \text{if } 1 \leq x \leq (t-1)n \\ x\alpha - j & \text{if } (t-1)n+1 \leq x \leq (t-1)n+k \\ n & \text{if } (t-1)n+k+1 \leq x \leq mn. \end{cases}$$

Then, $\beta \in M$ and $\alpha = \beta p_j$.

STEP 2. Now, in order to disregard the transformations p_ℓ, with $\ell > \lceil \frac{n-1}{2} \rceil$, for a given $j \in \{1, \dots, n-1\}$, we repeat STEP 1 considering, in particular, $\gamma = p_j$. As $1p_j = j+1$, we take

$$\alpha_j = p_j p_1^{n-(j+1)+1} = p_j p_1^{n-j} = \left(\begin{array}{cccccc|cc} 1 & \cdots & n-j & n-j+1 & \cdots & n & n+1 & \cdots \\ 1 & \cdots & n-j & n-j+1 & \cdots & n-j+1 & n-j+1 & \cdots \end{array}\right.$$
$$\left.\begin{array}{cc|ccccccc} \cdots & (m-1)n & (m-1)n+1 & \cdots & mn-j & mn-j+1 & \cdots & mn-1 & mn \\ \cdots & n-j+1 & n-j+1 & \cdots & n-j+1 & n-j+2 & \cdots & n & 1 \end{array}\right).$$

Notice that $\alpha_j \notin M$. Now, by STEP 1, there exists $\beta_j \in M$ such that $\alpha_j = \beta_j p_{(n-j+1)-1} = \beta_j p_{n-j}$. Thus, $p_j = \alpha_j p_1^j = \beta_j p_{n-j} p_1^j$, for some $\beta_j \in M$.

Finally, by noticing that $\lceil \frac{n-1}{2} \rceil < j \leq n-1$ implies $1 \leq n-j \leq \lceil \frac{n-1}{2} \rceil$, we may deduce that any transformation of $\mathcal{OP}_{mn,n}$ with rank n is a product of elements of $M \cup \{p_j \mid 1 \leq j \leq \lceil \frac{n-1}{2} \rceil\}$, as required. $\square$

Now, let

$$c_i = \left(\begin{array}{ccc|cccccc|ccc} 1 & \cdots & (i-1)n & (i-1)n+1 & (i-1)n+2 & (i-1)n+3 & \cdots & in & in+1 & \cdots & mn \\ 1 & \cdots & (i-1)n & (i-1)n+1 & (i-1)n+1 & (i-1)n+2 & \cdots & in-1 & in+1 & \cdots & mn \end{array}\right) \in \mathcal{O}_{m\times n}$$

and

$$b_{i,j} = \left(\begin{array}{ccc|ccccccc|} 1 & \cdots & (i-1)n & (i-1)n+1 & \cdots & (i-1)n+j-1 & (i-1)n+j & (i-1)n+j+1 & \cdots & in \\ 1 & \cdots & (i-1)n & (i-1)n+1 & \cdots & (i-1)n+j-1 & (i-1)n+j+1 & (i-1)n+j+1 & \cdots & in \end{array}\right.$$
$$\left.\begin{array}{|ccc} in+1 & \cdots & mn \\ in+1 & \cdots & mn \end{array}\right) \in \mathcal{O}_{m\times n},$$

for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n-1\}$; and

$$s_i = \left(\begin{array}{ccc|cccc|} 1 & \cdots & (i-1)n & (i-1)n+1 & (i-1)n+2 & \cdots & in \\ 1 & \cdots & (i-1)n & (i-1)n+1 & (i-1)n+1 & \cdots & (i-1)n+1 \end{array}\right.$$
$$\left.\begin{array}{|cccc|ccc} in+1 & in+2 & \cdots & (i+1)n & (i+1)n+1 & \cdots & mn \\ (i-1)n+1 & (i-1)n+2 & \cdots & in & (i+1)n+1 & \cdots & mn \end{array}\right) \in \mathcal{O}_{m\times n}$$

and

$$t_{i,j} = \left(\begin{array}{ccc|cccccc|} 1 & \cdots & (i-1)n & (i-1)n+1 & \cdots & in-j+1 & in-j+2 & \cdots & in \\ 1 & \cdots & (i-1)n & in+1 & \cdots & in+1 & in+2 & \cdots & in+j \end{array}\right.$$
$$\left.\begin{array}{|cccccc|ccc} in+1 & \cdots & in+j & in+j+1 & \cdots & (i+1)n & (i+1)n+1 & \cdots & mn \\ in+j & \cdots & in+j & in+j+1 & \cdots & (i+1)n & (i+1)n+1 & \cdots & mn \end{array}\right) \in \mathcal{O}_{m\times n},$$

for $i \in \{1, \dots, m-1\}$ and $j \in \{1, \dots, n\}$. The authors proved in [12, Proposition 2.5] that the set

$$\{c_i, b_{i,j}, s_k, t_{k,\ell} \mid 1 \leq i \leq m, 1 \leq j \leq n-1, 1 \leq k \leq m-1, 2 \leq \ell \leq n\}$$

is a generating set of the monoid $\mathcal{O}_{m\times n}$. On the other hand, it is a routine matter to show that:

1. $c_i = f^{m-i+1}c_1f^{i-1}$, for $2 \le i \le m$;
2. $b_{i,j} = f^{m-i+1}b_{1,j}f^{i-1}$, for $2 \le i \le m$ and $1 \le j \le n-1$;
3. $s_i = f^{m-i+1}s_1f^{i-1}$, for $2 \le i \le m-1$; and
4. $t_{i,j} = f^{m-i+1}t_{1,j}f^{i-1}$, for $2 \le i \le m-1$ and $2 \le j \le n$.

These observations combined with the previous three lemmas, allow us to deduce the following result.

Proposition 2.5 *The set* $A = \{f, c_1, b_{1,1}\ldots, b_{1,n-1}, s_1, t_{1,2}\ldots, t_{1,n}, p_1, \ldots, p_{\lceil\frac{n-1}{2}\rceil}\}$ *is a generating set, with* $2n+\lceil\frac{n-1}{2}\rceil+1$ *elements, of the monoid* $\mathcal{OP}_{m\times n}$.

Example 2.1 The monoid $\mathcal{OP}_{3\times 4}$ is generated by the following transformations:

$$f = \left(\begin{array}{cccc|cccc|cccc} 1&2&3&4&5&6&7&8&9&10&11&12\\ 5&6&7&8&9&10&11&12&1&2&3&4 \end{array}\right); \quad c_1 = \left(\begin{array}{cccc|cccc|cccc} 1&2&3&4&5&6&7&8&9&10&11&12\\ 1&1&2&3&5&6&7&8&9&10&11&12 \end{array}\right);$$

$$b_{1,1} = \left(\begin{array}{cccc|cccc|cccc} 1&2&3&4&5&6&7&8&9&10&11&12\\ 2&2&3&4&5&6&7&8&9&10&11&12 \end{array}\right); \quad s_1 = \left(\begin{array}{cccc|cccc|cccc} 1&2&3&4&5&6&7&8&9&10&11&12\\ 1&1&1&1&1&2&3&4&9&10&11&12 \end{array}\right);$$

$$b_{1,2} = \left(\begin{array}{cccc|cccc|cccc} 1&2&3&4&5&6&7&8&9&10&11&12\\ 1&3&3&4&5&6&7&8&9&10&11&12 \end{array}\right); \quad t_{1,2} = \left(\begin{array}{cccc|cccc|cccc} 1&2&3&4&5&6&7&8&9&10&11&12\\ 5&5&5&6&6&6&7&8&9&10&11&12 \end{array}\right);$$

$$b_{1,3} = \left(\begin{array}{cccc|cccc|cccc} 1&2&3&4&5&6&7&8&9&10&11&12\\ 1&2&4&4&5&6&7&8&9&10&11&12 \end{array}\right); \quad t_{1,3} = \left(\begin{array}{cccc|cccc|cccc} 1&2&3&4&5&6&7&8&9&10&11&12\\ 5&5&6&7&7&7&7&8&9&10&11&12 \end{array}\right);$$

$$p_1 = \left(\begin{array}{cccc|cccc|cccc} 1&2&3&4&5&6&7&8&9&10&11&12\\ 2&3&4&1&1&1&1&1&1&1&1&2 \end{array}\right); \quad t_{1,4} = \left(\begin{array}{cccc|cccc|cccc} 1&2&3&4&5&6&7&8&9&10&11&12\\ 5&6&7&8&8&8&8&8&9&10&11&12 \end{array}\right);$$

$$p_2 = \left(\begin{array}{cccc|cccc|cccc} 1&2&3&4&5&6&7&8&9&10&11&12\\ 3&4&1&1&1&1&1&1&1&1&2&3 \end{array}\right).$$

The next series of lemmas will allow us to conclude that A is a least size generating set of $\mathcal{OP}_{m\times n}$, for $m > 2$, and contains a least size generating set of $\mathcal{OP}_{2\times n}$.

Lemma 2.6 *Any generating set of* $\mathcal{OP}_{m\times n}$ *contains at least a non identity element of rank* mn *and* n *distinct elements of rank* $mn-1$.

Proof. Let X be a generating set of $\mathcal{OP}_{m\times n}$.

Notice that, the group of units of $\mathcal{OP}_{m\times n}$ is, clearly, generated by the permutation $f = g^n$, which has order m. Hence, X must contain a non identity element of rank mn.

Now, let $\xi_1, \ldots, \xi_k$ ($k \ge 1$) be all the elements of X of rank $mn-1$. Then, any element of $\mathcal{OP}_{m\times n}$ of rank $mn-1$ is of the form $\alpha\xi_jf^i$, for some $j \in \{1, \ldots, k\}$, $i \in \{0, \ldots, m-1\}$ and a product $\alpha \in \mathcal{OP}_{m\times n}$. As the elements of rank $mn-1$ of the above form can not have more than mk distinct images and, on the other hand, we have precisely mn possible distinct images for an element of $\mathcal{OP}_{m\times n}$ of rank $mn-1$, we deduce that $mn \le mk$ and so $k \ge n$, as required. □

Lemma 2.7 *For* $m > 2$, *any generating set of* $\mathcal{OP}_{m\times n}$ *contains at least* n *distinct elements of rank* $(m-1)n$.

Proof. Let $T_j = \{\alpha \in \mathcal{OP}_{m\times n} \mid \mathrm{rank}(\alpha) = (m-1)n \text{ and } (kn)\alpha = (kn+1)\alpha = (i-1)n+j, \text{ for some } 1 \le i, k \le m\}$, for $j \in \{1, \ldots, n\}$. Clearly, $T_1, \ldots, T_n$ are n two by two disjoint non-empty subsets of elements of rank $(m-1)n$ of $\mathcal{OP}_{m\times n}$.

Let $j \in \{1, \ldots, n\}$ and take $\alpha \in T_j$. Let $i, k \in \{1, \ldots, m\}$ be such that $(kn)\alpha = (kn+1)\alpha = (i-1)n+j$.

Suppose that $\alpha = \alpha'\alpha''$, for some $\alpha', \alpha'' \in \mathcal{OP}_{m\times n}$, and take $\alpha\psi = (\alpha_1, \ldots, \alpha_m; \beta)$, $\alpha'\psi = (\alpha'_1, \ldots, \alpha'_m; \beta')$ and $\alpha''\psi = (\alpha''_1, \ldots, \alpha''_m; \beta'')$. Notice that, α, α' and α'' are elements of $\mathcal{OP}_{m\times n}$ of type 1 (whence $\alpha_\ell, \alpha'_\ell, \alpha''_\ell \in \mathcal{O}_n$, for $\ell \in \{1, \ldots, m\}$). Also, observe that $n\alpha_k = j = 1\alpha_{k+1}$, $\mathrm{Im}(\alpha_k) = \{1, \ldots, j\}$, $\mathrm{Im}(\alpha_{k+1}) = \{j, \ldots, n\}$ and $\alpha_\ell = 1$, for $\ell \in \{1, \ldots, m\} \setminus \{k, k+1\}$. Moreover, we have $\alpha_\ell = \alpha'_\ell\alpha''_{\ell\beta'}$, for $\ell \in \{1, \ldots, m\}$, and, on the other hand, $\beta = \beta'\beta''$, from which follows that $\mathrm{rank}(\beta') = m-1$ or $\mathrm{rank}(\beta'') = m-1$, since $\mathrm{rank}(\beta) = m-1$.

Next, our goal is to show that $\alpha' \in T_j$ or $\alpha'' \in T_j$. We consider two cases: $\mathrm{rank}(\beta') = m-1$ or $\mathrm{rank}(\beta') = m$.

First, admit that β' has rank $m-1$. Then α' has rank $(m-1)n$ and so $\mathrm{Ker}(\alpha') = \mathrm{Ker}(\alpha)$. Hence $\mathrm{Ker}(\alpha'_k) = \mathrm{Ker}(\alpha_k)$ and $\mathrm{Ker}(\alpha'_{k+1}) = \mathrm{Ker}(\alpha_{k+1})$, from which follows that $|\mathrm{Im}(\alpha'_k)| = |\mathrm{Im}(\alpha_k)| = j$ and $|\mathrm{Im}(\alpha'_{k+1})| = |\mathrm{Im}(\alpha_{k+1})| = n-j+1$. Thus $n\alpha'_k \ge j$ and $1\alpha'_{k+1} \le j$, since $1\alpha'_k \le \cdots \le n\alpha'_k$ and $1\alpha'_{k+1} \le \cdots \le n\alpha'_{k+1}$. On the other hand, the equality $\mathrm{Ker}(\alpha') = \mathrm{Ker}(\alpha)$ also implies that $(kn)\alpha' = (kn+1)\alpha'$, whence $n\alpha'_k = 1\alpha'_{k+1}$ and so $n\alpha'_k = 1\alpha'_{k+1} = j$. Then $(kn)\alpha' = (kn+1)\alpha' = (k\beta'-1)n+j$ and, finally, we conclude that $\alpha' \in T_j$.

Secondly, suppose that β' has rank m (i.e. β' is a power of g_m). Then β'' must have rank $m-1$ and so α'' has rank $(m-1)n$. As $\alpha'_k\alpha''_{k\beta'} = \alpha_k$, then $\{1,\dots,j\} = \mathrm{Im}(\alpha_k) \subseteq \mathrm{Im}(\alpha''_{k\beta'})$ and so $n\alpha''_{k\beta'} \geq j$. Similarly, as $\alpha'_{k+1}\alpha''_{(k+1)\beta'} = \alpha_{k+1}$, then $\{j,\dots,n\} = \mathrm{Im}(\alpha_{k+1}) \subseteq \mathrm{Im}(\alpha''_{(k+1)\beta'})$ and so $1\alpha''_{(k+1)\beta'} \leq j$. Now, by noticing that β' is a power a g_m, we have $(k+1)\beta' = k\beta'+1$ and so $(k\beta'+1)\beta'' = ((k+1)\beta')\beta'' = (k+1)\beta = k\beta = (k\beta')\beta''$. Hence, $j \leq n\alpha''_{k\beta'} \leq 1\alpha''_{k\beta'+1} = 1\alpha''_{(k+1)\beta'} \leq j$, i.e. $n\alpha''_{k\beta'} = 1\alpha''_{k\beta'+1} = j$, from which follows that $((k\beta')n)\alpha'' = ((k\beta')n+1)\alpha'' = (k\beta-1)n+j$. Thus $\alpha'' \in T_j$, as required.

Now, by induction on k, it is clear that to write an element of T_j as a product of k elements of $\mathcal{OP}_{m\times n}$, we must have a factor belonging to T_j, for all $1 \leq j \leq n$. This fact proves the lemma. $\square$

The next lemma helps us to find the least number of elements of rank n required on a generating set of $\mathcal{OP}_{m\times n}$.

Lemma 2.8 *Let $(\alpha_1,\dots,\alpha_m;\beta) \in \mathcal{OP}_{m\times n}\psi$ be such that $i\beta = j\beta$, for some $1 \leq i < j \leq m$. Then $|\mathrm{Im}(\alpha_i)| + |\mathrm{Im}(\alpha_j)| \leq n+2$. Moreover, if $|\mathrm{Im}(\alpha_i)| + |\mathrm{Im}(\alpha_j)| = n+2$, then $(\alpha_1,\dots,\alpha_m;\beta)$ is of type 3 and:*

1. *$\alpha_i \in \mathcal{OP}_n \setminus \mathcal{O}_n$ or $\alpha_j \in \mathcal{OP}_n \setminus \mathcal{O}_n$;*
2. *$\mathrm{Im}(\alpha_i) \cup \mathrm{Im}(\alpha_j) = A_{i\beta}$ (and so $(\alpha_1,\dots,\alpha_m;\beta)\psi^{-1}$ is a transformation of $\mathcal{OP}_{m\times n}$ of rank n);*
3. *$|\mathrm{Im}(\alpha_k)| = 1$, for $k \in \{1,\dots,m\} \setminus \{i,j\}$.*

Proof. We begin by proving that $|\mathrm{Im}(\alpha_i)| + |\mathrm{Im}(\alpha_j)| \leq n+2$.

First, suppose that $\alpha_i, \alpha_j \in \mathcal{O}_n$. Then, as $i\beta = j\beta$, we have

$$1\alpha_i \leq \cdots \leq n\alpha_i \leq 1\alpha_j \leq \cdots \leq n\alpha_j \quad \text{or} \quad 1\alpha_j \leq \cdots \leq n\alpha_j \leq 1\alpha_i \leq \cdots \leq n\alpha_i,$$

whence $\mathrm{Im}(\alpha_i) \cup \mathrm{Im}(\alpha_j)$ has at least $|\mathrm{Im}(\alpha_i)| + |\mathrm{Im}(\alpha_j)| - 1$ distinct elements (notice that we may have $n\alpha_i = 1\alpha_j$ or $n\alpha_j = 1\alpha_i$). As $\mathrm{Im}(\alpha_i) \cup \mathrm{Im}(\alpha_j) \subseteq A_{i\beta}$, it follows that $|\mathrm{Im}(\alpha_i)| + |\mathrm{Im}(\alpha_j)| \leq n+1$.

Next, suppose that $\alpha_i \in \mathcal{OP}_n \setminus \mathcal{O}_n$. Then $\alpha_j \in \mathcal{O}_n$ and we have

$$(t+1)\alpha_i \leq \cdots \leq n\alpha_i \leq 1\alpha_j \leq \cdots \leq n\alpha_j \leq 1\alpha_i \leq \cdots \leq t\alpha_i,$$

for some $1 \leq t \leq m-1$, whence $\mathrm{Im}(\alpha_i) \cup \mathrm{Im}(\alpha_j)$ has at least $|\mathrm{Im}(\alpha_i)| + |\mathrm{Im}(\alpha_j)| - 2$ distinct elements (notice that we may have $n\alpha_i = 1\alpha_j$ and $n\alpha_j = 1\alpha_i$) and so $|\mathrm{Im}(\alpha_i)| + |\mathrm{Im}(\alpha_j)| \leq n+2$.

Finally, as the case $\alpha_j \in \mathcal{OP}_n \setminus \mathcal{O}_n$ is similar to the previous one, we proved that $|\mathrm{Im}(\alpha_i)| + |\mathrm{Im}(\alpha_j)| \leq n+2$, for all cases.

Now, in order to prove the second part of the lemma, admit that $|\mathrm{Im}(\alpha_i)| + |\mathrm{Im}(\alpha_j)| = n+2$.

Notice that, by the first part of the proof, $\alpha_i \in \mathcal{OP}_n \setminus \mathcal{O}_n$ or $\alpha_j \in \mathcal{OP}_n \setminus \mathcal{O}_n$ (and so $(\alpha_1,\dots,\alpha_m;\beta)$ must be of type 3). On the other hand, as $n \geq |\mathrm{Im}(\alpha_i) \cup \mathrm{Im}(\alpha_j)| \geq |\mathrm{Im}(\alpha_i)| + |\mathrm{Im}(\alpha_j)| - 2 = n$, we have $\mathrm{Im}(\alpha_i) \cup \mathrm{Im}(\alpha_j) = A_{i\beta}$.

Suppose that $\alpha_i \in \mathcal{OP}_n \setminus \mathcal{O}_n$ and let $t \in \{1,\dots,m-1\}$ be as above. Let $k \in \{i+1,\dots,j-1\}$. Let $\ell \in \{1,\dots,m\} \setminus \{i,\dots,j\}$. Then (with the obvious adaptation if k or ℓ does not exist) we have

$$(t+1)\alpha_i \leq \cdots \leq n\alpha_i \leq 1\alpha_k \leq \cdots \leq n\alpha_k \leq 1\alpha_j \leq \cdots \leq n\alpha_j \leq 1\alpha_\ell \leq \cdots \leq n\alpha_\ell \leq 1\alpha_i \leq \cdots \leq t\alpha_i.$$

Hence, $n\alpha_i = 1\alpha_j$ and $n\alpha_j = 1\alpha_i$, otherwise $\mathrm{Im}(\alpha_i) \cup \mathrm{Im}(\alpha_j)$ would have at least $|\mathrm{Im}(\alpha_i)| + |\mathrm{Im}(\alpha_j)| - 1 = n+1$ distinct elements, which is a contradiction. Thus, $n\alpha_i = 1\alpha_k = \cdots = n\alpha_k = 1\alpha_j$ (if k exists) and $n\alpha_j = 1\alpha_\ell = \cdots = n\alpha_\ell = 1\alpha_i$ (if ℓ exists). Anyway, we proved that $|\mathrm{Im}(\alpha_k)| = 1$, for $k \in \{1,\dots,m\} \setminus \{i,j\}$.

Similarly, if $\alpha_j \in \mathcal{OP}_n \setminus \mathcal{O}_n$, we have $|\mathrm{Im}(\alpha_k)| = 1$, for $k \in \{1,\dots,m\} \setminus \{i,j\}$, as required. $\square$

Lemma 2.9 *Any generating set of $\mathcal{OP}_{m\times n}$ contains at least $\lceil \frac{n-1}{2} \rceil$ elements of rank n.*

Proof. For $1 \leq i \leq \lceil \frac{n-1}{2} \rceil$, define

$$P_i = \{(\gamma_1,\dots,\gamma_m;\lambda) \in \overline{N} \mid |\mathrm{Im}(\gamma_k)| = n-i+1 \text{ and } |\mathrm{Im}(\gamma_\ell)| = i+1, \text{ for some } 1 \leq k,\ell \leq m \text{ such that } k \neq \ell\}.$$

Notice that $p_i\psi \in P_i$ and, by Lemma 2.8, all elements of P_i are of type 3 and all elements of $P_i\psi^{-1}$ have rank n, for $1 \leq i \leq \lceil \frac{n-1}{2} \rceil$. Moreover, $P_1,\dots,P_{\lceil \frac{n-1}{2} \rceil}$ are $\lceil \frac{n-1}{2} \rceil$ two by two disjoint subsets of $\mathcal{OP}_{m\times n}\psi$. In fact, suppose there exists $(\gamma_1,\dots,\gamma_m;\lambda) \in P_i \cap P_j$, for some $1 \leq i < j \leq \lceil \frac{n-1}{2} \rceil$. Let $1 \leq k,\ell \leq m$, with $k \neq \ell$, be such that $|\mathrm{Im}(\gamma_k)| = n-i+1$ and $|\mathrm{Im}(\gamma_\ell)| = i+1$. Then, by Lemma 2.8, we have $|\mathrm{Im}(\gamma_t)| = 1$, for $t \in \{1,\dots,m\} \setminus \{k,\ell\}$. Hence $|\mathrm{Im}(\gamma_k)| = n-j+1$ or $|\mathrm{Im}(\gamma_k)| = j+1$. If $|\mathrm{Im}(\gamma_k)| = n-j+1$ then $i = j$, which is a contradiction. On the other hand, if $|\mathrm{Im}(\gamma_k)| = j+1$ then $n = i+j < \lceil \frac{n-1}{2} \rceil + \lceil \frac{n-1}{2} \rceil \leq \frac{n}{2} + \frac{n}{2} = n$, which is again a contradiction. Therefore, $P_i \cap P_j = \emptyset$, for $1 \leq i < j \leq \lceil \frac{n-1}{2} \rceil$.

It follows that $P_1\psi^{-1},\dots,P_{\lceil\frac{n-1}{2}\rceil}\psi^{-1}$ are $\lceil\frac{n-1}{2}\rceil$ two by two disjoint subsets of $\mathcal{OP}_{m\times n}$ of elements of rank n.

Now, let $i\in\{1,\dots,\lceil\frac{n-1}{2}\rceil\}$ and take $\overline{\gamma}=(\gamma_1,\dots,\gamma_m;\lambda)\in P_i$. Let $1\le k,\ell\le m$, with $k\ne\ell$, be such that $|\operatorname{Im}(\gamma_k)|=n-i+1$ and $|\operatorname{Im}(\gamma_\ell)|=i+1$.

Suppose that $\overline{\gamma}=\alpha\psi\alpha'\psi$, for some $\alpha,\alpha'\in\mathcal{OP}_{m\times n}$, and take $\alpha\psi=(\alpha_1,\dots,\alpha_m;\beta)$ and $\alpha'\psi=(\alpha'_1,\dots,\alpha'_m;\beta')$. Notice that $\gamma_j=\alpha_j\alpha'_{j\beta}$, for $1\le j\le m$. Moreover, as $\overline{\gamma}$ is of type 3, then either α or α' is of type 3.

Next, our goal is to show that $\alpha\psi\in P_i$ or $\alpha'\psi\in P_i$.

We begin by observing that if $k\beta=\ell\beta$ then, by Lemma 2.8, we have $|\operatorname{Im}(\alpha_k)|+|\operatorname{Im}(\alpha_\ell)|\le n+2$ and so, as

$$n-i+1=|\operatorname{Im}(\gamma_k)|=|(\operatorname{Im}(\alpha_k))\alpha'_{k\beta}|\le|\operatorname{Im}(\alpha_k)|\quad\text{and}\quad i+1=|\operatorname{Im}(\gamma_\ell)|=|(\operatorname{Im}(\alpha_\ell))\alpha'_{\ell\beta}|\le|\operatorname{Im}(\alpha_\ell)|,$$

it follows that $|\operatorname{Im}(\alpha_k)|=n-i+1$ and $|\operatorname{Im}(\alpha_\ell)|=i+1$. We consider two cases.

First, if α is of type 3 (in particular, we have $\alpha\psi\in\overline{N}$), as β is a constant transformation, we have $k\beta=\ell\beta$ and so, by the above observation, we may deduce immediately that $\alpha\psi\in P_i$.

On the other hand, admit that α is not of type 3. Then $k\beta\ne\ell\beta$. In fact, if $k\beta=\ell\beta$ then $|\operatorname{Im}(\alpha_k)|+|\operatorname{Im}(\alpha_\ell)|=(n-i+1)+(i+1)=n+2$ and so, by Lemma 2.8, α must be of type 3, which is a contradiction. Also, notice that α' must be of type 3 and so β' is a constant transformation. In particular, $(k\beta)\beta'=(\ell\beta)\beta'$. Hence, by Lemma 2.8, we have $|\operatorname{Im}(\alpha'_{k\beta})|+|\operatorname{Im}(\alpha'_{\ell\beta})|\le n+2$. Moreover, since $\operatorname{Im}(\gamma_k)=\operatorname{Im}(\alpha_k\alpha'_{k\beta})\subseteq\operatorname{Im}(\alpha'_{k\beta})$ and $\operatorname{Im}(\gamma_\ell)=\operatorname{Im}(\alpha_\ell\alpha'_{\ell\beta})\subseteq\operatorname{Im}(\alpha'_{\ell\beta})$, we have $n-i+1=|\operatorname{Im}(\gamma_k)|\le|\operatorname{Im}(\alpha'_{k\beta})|$ and $i+1=|\operatorname{Im}(\gamma_\ell)|\le|\operatorname{Im}(\alpha'_{\ell\beta})|$. Thus, we obtain precisely $|\operatorname{Im}(\alpha'_{k\beta})|=n-i+1$ and $|\operatorname{Im}(\alpha'_{\ell\beta})|=i+1$, which proves that $\alpha'\psi\in P_i$.

Now, by induction on k, it is easy to show that to write an element of $P_i\psi^{-1}$ as a product of k elements of $\mathcal{OP}_{m\times n}$, we must have a factor that belongs to $P_i\psi^{-1}$, for all $1\le i\le\lceil\frac{n-1}{2}\rceil$. This fact proves the lemma. □

Now, for $m>2$, from the previous lemmas, we deduce immediately that A is a least size generating set of $\mathcal{OP}_{m\times n}$. On the other and, regarding $\mathcal{OP}_{2\times n}$, it is a routine matter to show that:

1. $s_1=b_{1,1}b_{1,2}\cdots b_{1,n-1}fp_1^n$;
2. $t_{1,j}=fc_1^{j-1}p_{j-1}f$, for $2\le j\le\lceil\frac{n-1}{2}\rceil+1$
3. $t_{1,j}=c_1^{n-1-j}b_{1,1}p_{n-j}{p_1}^{j-1}f$, for $\lceil\frac{n-1}{2}\rceil+2\le j\le n-2$; and
4. $t_{1,n-1}=b_{1,1}p_1^{n-1}f$ and $t_{1,n}=fc_1fp_1^nf$.

Hence, from these equalities and Lemmas 2.6 and 2.9, it follows that $\{f,c_1,b_{1,1}\dots,b_{1,n-1},p_1,\dots,p_{\lceil\frac{n-1}{2}\rceil}\}$ is a least size generating set of $\mathcal{OP}_{2\times n}$. Therefore, we have proved:

Theorem 2.10 *The rank of $\mathcal{OP}_{m\times n}$ is equal to $2n+\lceil\frac{n-1}{2}\rceil+1$, for $m>2$, and equal to $n+\lceil\frac{n-1}{2}\rceil+1$, for $m=2$.*

3 The rank of the monoid $\mathcal{OD}_{m\times n}$

Consider the reflexion

$$h=\begin{pmatrix}1&2&\cdots&n-1&mn\\ mn&mn-1&\cdots&2&1\end{pmatrix}.$$

Observe that h is a permutation of order two of X_{mn} and $h\in\mathcal{OD}_{m\times n}$. Moreover, given $\alpha\in\mathcal{T}_{mn}$, we have $\alpha=h^2\alpha=h(h\alpha)$ and α is an order-reversing transformation if and only if $h\alpha$ (respectively, αh) is an order-preserving transformation. Thus, clearly, the monoid $\mathcal{OD}_{m\times n}$ is generated by $\mathcal{O}_{m\times n}\cup\{h\}$. As recalled in Section 2, the authors proved in [12] that

$$C=\{c_i,b_{i,j},s_k,t_{k,\ell}\mid 1\le i\le m,1\le j\le n-1,1\le k\le m-1,2\le\ell\le n\}$$

is a generating set, with $2mn-n$ elements, of the monoid $\mathcal{O}_{m\times n}$. Hence $C\cup\{h\}$ generates $\mathcal{OD}_{m\times n}$. In order to reduce the number of generators, consider

$$s_{i,j}=\left(\begin{array}{ccc|cccccc}1&\cdots&(i-1)n&(i-1)n+1&\cdots&in-j+1&in-j+2&\cdots&in\\ 1&\cdots&(i-1)n&(i-1)n+1&\cdots&in-j+1&in-j+1&\cdots&in-j+1\end{array}\right.$$
$$\left.\begin{array}{|cccccc|ccc}in+1&in+2&\cdots&in+j&\cdots&(i+1)n&(i+1)n+1&\cdots&mn\\ in-j+1&in-j+2&\cdots&in&\cdots&in&(i+1)n+1&\cdots&mn\end{array}\right)\in\mathcal{O}_{m\times n},$$

for $i\in\{1,\dots,m-1\}$ and $j\in\{1,\dots,n\}$. Notice that, we have $s_i=s_{i,n}$, for $i\in\{1,\dots,m-1\}$. Finally, if m is odd, consider also the transformation $u_j\in\mathcal{O}_{m\times n}$ of rank $mn-1$, whose image is $\{1,\dots,mn\}\setminus\{\frac{m-1}{2}n+j\}$ and whose kernel is defined by the partition $\{\{1\},\dots,\{\frac{m-1}{2}n+\lceil\frac{n}{2}\rceil-j\},\{\frac{m-1}{2}n+\lceil\frac{n}{2}\rceil-j+1,\frac{m-1}{2}n+\lceil\frac{n}{2}\rceil-j+2\},\{\frac{m-1}{2}n+\lceil\frac{n}{2}\rceil-j+3\},\dots,\{mn\}\}$, for each $1\le j\le\lceil\frac{n}{2}\rceil$.

Example 3.1 For $m=3$ and $n=5$, we have:

$$u_1=\left(\begin{array}{ccccc|ccccc|ccccc} 1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ 1&2&3&4&5&7&8&9&9&10&11&12&13&14&15 \end{array}\right),$$

$$u_2=\left(\begin{array}{ccccc|ccccc|ccccc} 1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ 1&2&3&4&5&6&8&8&9&10&11&12&13&14&15 \end{array}\right),$$

$$u_3=\left(\begin{array}{ccccc|ccccc|ccccc} 1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ 1&2&3&4&5&6&6&7&9&10&11&12&13&14&15 \end{array}\right).$$

The proofs of the equalities stated in the next lemma are routine.

Lemma 3.1 *The following identities hold:*

1. $c_i = hb_{m-i+1,n-1}b_{m-i+1,n-2}\cdots b_{m-i+1,1}h$, *for* $1\le i\le m$;
2. $b_{i,1} = hb_{m-i+1,n-1}b_{m-i+1,n-2}\cdots b_{m-i+1,2}c_{m-i+1}h$, *for* $1\le i\le m$;
3. $b_{i,j} = hb_{m-i+1,n-j}b_{m-i+1,n-j-1}\cdots b_{m-i+1,2}c_{m-i+1}b_{m-i+1,n-1}b_{m-i+1,n-2}\cdots b_{m-i+1,n-j+1}h$, *for* $1\le i\le m$ *and* $2\le j\le n-1$;
4. $b_{\frac{m+1}{2},j} = u_{\lceil\frac{n}{2}\rceil-j+1}u_j$, *for m odd and* $1\le j\le\lceil\frac{n}{2}\rceil$;
5. $b_{\frac{m+1}{2},n-j} = hu_{\lceil\frac{n}{2}\rceil-j+1}u_{j+1}h$, *for m odd and* $1\le j\le n-\lceil\frac{n}{2}\rceil-1$;
6. $t_{i,j} = hs_{m-i,j}h$, *for* $1\le i\le m-1$ *and* $1\le j\le n$;
7. $t_{i,j} = c_i^{n-j}hb'_2\cdots b'_jhs_{i,n-j+1}hs_{m-i}h$, *with* $b'_\ell = b_{m-i,n-j+\ell-1}b_{m-i,n-j+\ell-2}\cdots b_{m-i,\ell}$, *for* $2\le\ell\le j$, $1\le i\le m-1$ *and* $2\le j\le n-1$. $\square$

Now, let A be the set

$$\{c_i,b_{i,j},s_{k,\ell},s_{\frac{m}{2},r},s_t,h\mid 1\le i\le\frac{m}{2},\ 1\le j\le n-1,\ 1\le k\le\frac{m}{2}-1,\ 2\le\ell\le n-1,\ n-\left\lceil\frac{n}{2}\right\rceil+1\le r\le n-1,\ 1\le t\le m-1\},$$

if m is even, and the set

$$\{c_i,b_{i,j},u_k,s_{i,\ell},s_t,h\mid 1\le i\le\frac{m-1}{2},\ 1\le j\le n-1,\ 1\le k\le\left\lceil\frac{n}{2}\right\rceil,\ 2\le\ell\le n-1,\ 1\le t\le m-1\},$$

if m is odd. By using the equalities of Lemma 3.1, it is not difficult to show that any element of C is a product of elements of A. Thus, it follows that:

Proposition 3.2 *The set A generates the monoid $\mathcal{OD}_{m\times n}$. Furthermore, A has* $\lceil\frac{mn}{2}\rceil+\left\lceil\frac{(m-1)n}{2}\right\rceil+1$ *elements.* $\square$

Next, we aim to show that the rank of $\mathcal{OD}_{m\times n}$ is precisely $\lceil\frac{mn}{2}\rceil+\left\lceil\frac{(m-1)n}{2}\right\rceil+1$.

Let U be a generating set of $\mathcal{OD}_{m\times n}$.

First, notice that, as h is the unique non-identity permutation in $\mathcal{OD}_{m\times n}$, we must have $h\in U$.

On the other hand, recall that, in the proof of [9, Theorem 1.5], Fernandes et al. showed that any generating set of the monoid $\mathcal{OD}_n$, for $n\ge 2$, has at least $\lceil\frac{n}{2}\rceil$ elements of rank $n-1$. A similar argument allow us to conclude that U must have at least $\lceil\frac{mn}{2}\rceil$ elements of rank $mn-1$. In fact, take $K_i=\{1,2,\ldots,mn\}\setminus\{i\}$, for $1\le i\le mn$, and let $\xi_1,\ldots,\xi_k$ be all the elements of U of rank $mn-1$. Then $k\ge 1$ and, for all $1\le i\le k$, there exists $1\le\ell_i\le mn$ such that $\operatorname{Im}(\xi_i)=K_{\ell_i}$. Now, given an element $\alpha\in\mathcal{OD}_{m\times n}$ of rank $mn-1$, we have $\alpha=\xi\xi_i$ or $\alpha=\xi\xi_ih$, for some $\xi\in\mathcal{OD}_{m\times n}$ and $1\le i\le k$. Hence, $\operatorname{Im}(\alpha)=\operatorname{Im}(\xi_i)=K_{\ell_i}$ or $\operatorname{Im}(\alpha)=\operatorname{Im}(\xi_ih)=K_{mn-\ell_i+1}$. As we have mn possible distinct images for a transformation of $\mathcal{OD}_{m\times n}$ of rank $mn-1$, the set $\{K_{\ell_1},\ldots,K_{\ell_k},K_{mn-\ell_1+1},\ldots,K_{mn-\ell_k+1}\}$ has at least mn elements. It follows that $2k\ge mn$ and so $k\ge\lceil\frac{mn}{2}\rceil$.

Therefore, we have proved:

Lemma 3.3 *Any generating set of $\mathcal{OD}_{m\times n}$ contains h and at least* $\lceil\frac{mn}{2}\rceil$ *distinct elements of rank* $mn-1$. $\square$

Regarding generators of rank $(m-1)n$, we have:

Lemma 3.4 *Any generating set of $\mathcal{OD}_{m\times n}$ contains at least $\left\lceil\frac{(m-1)n}{2}\right\rceil$ distinct elements of rank $(m-1)n$.*

Proof. For $1\le i\le m-1$ and $1\le j\le n$, consider

$$Q_{i,j}=\{\alpha\in\mathcal{O}_{m\times n}\mid \mathrm{rank}(\alpha)=(m-1)n \text{ and } (in)\alpha=(in+1)\alpha=(k-1)n+j, \text{ for some } 1\le k\le m\}.$$

The authors proved in [12, Theorem 2.6] that the family $\{Q_{i,j}\mid 1\le i\le m-1, 1\le j\le n\}$ consists on $(m-1)n$ two by two disjoint non-empty subsets of $\mathcal{O}_{m\times n}$ such that, given $\alpha_1,\alpha_2\in\mathcal{O}_{m\times n}$, if $\alpha_1\alpha_2\in Q_{i,j}$ then $\alpha_1\in Q_{i,j}$ or $\alpha_2\in Q_{i,j}$, for $1\le i\le m-1$ and $1\le j\le n$. On the other hand, given $\alpha\in\mathcal{T}_{mn}$, it is easy to show that

$$\alpha\in Q_{i,j} \text{ if and only if } h\alpha h\in Q_{m-i,n-j+1} \quad \text{and, consequently,} \quad h\alpha\in Q_{i,j} \text{ if and only if } \alpha h\in Q_{m-i,n-j+1}, \qquad (3)$$

for $1\le i\le m-1$ and $1\le j\le n$.

Next, for $1\le i\le m-1$ and $1\le j\le n$, define

$$T_{i,j}=\{\alpha\in\mathcal{OD}_{m\times n}\mid \alpha\in Q_{i,j}\cup Q_{m-i,n-j+1} \text{ or } h\alpha\in Q_{i,j}\cup Q_{m-i,n-j+1}\}.$$

Observe that, clearly, $T_{i,j}=T_{m-i,n-j+1}$, for $1\le i\le m-1$ and $1\le j\le n$. Moreover, if $1\le i,i'\le m-1$ and $1\le j,j'\le n$ are such that $T_{i,j}\cap T_{i',j'}\neq\emptyset$ then $(i',j')=(i,j)$ or $(i',j')=(m-i,n-j+1)$. In fact, suppose that there exists $\alpha\in T_{i,j}\cap T_{i',j'}$. If $\alpha\in\mathcal{O}_{m\times n}$ then $\alpha\in(Q_{i,j}\cup Q_{m-i,n-j+1})\cap(Q_{i',j'}\cup Q_{m-i',n-j'+1})$. On the other hand, if $\alpha\notin\mathcal{O}_{m\times n}$ then $h\alpha\in(Q_{i,j}\cup Q_{m-i,n-j+1})\cap(Q_{i',j'}\cup Q_{m-i',n-j'+1})$. Then, for both cases, $Q_{i,j}\cap Q_{i',j'}\neq\emptyset$ or $Q_{i,j}\cap Q_{m-i',n-j'+1}\neq\emptyset$ or $Q_{m-i,n-j+1}\cap Q_{i',j'}\neq\emptyset$ or $Q_{m-i,n-j+1}\cap Q_{m-i',n-j'+1}\neq\emptyset$, from which follows that $(i',j')=(i,j)$ or $(i',j')=(m-i,n-j+1)$, regarding that $\{Q_{i,j}\mid 1\le i\le m-1, 1\le j\le n\}$ has $(m-1)n$ two by two disjoint elements.

Therefore, we may deduce that the family $\{T_{i,j}\mid 1\le i\le m-1, 1\le j\le n\}$ consists on $\left\lceil\frac{(m-1)n}{2}\right\rceil$ two by two disjoint non-empty subsets of $\mathcal{OD}_{m\times n}$.

Now, by proving that any generating set of $\mathcal{OD}_{m\times n}$ contains an element of $T_{i,j}$, for all $1\le i\le m-1$ and $1\le j\le n$, the proof of the lemma follows. To accomplish this aim, we show that, for $1\le i\le m-1$ and $1\le j\le n$, given $\alpha_1,\ldots,\alpha_k\in\mathcal{OD}_{m\times n}$ ($k\in\mathbb{N}$) such that $\alpha_1\cdots\alpha_k\in T_{i,j}$, we have $\alpha_t\in T_{i,j}$, for some $t\in\{1,\ldots,k\}$. Furthermore, in order to prove this last statement, by induction on k, it suffices to consider $k=2$.

First, notice that, given $\alpha\in\mathcal{T}_{mn}$, it follows from (3) that

$$\{\alpha,h\alpha h,\alpha h,h\alpha\}\subseteq T_{i,j} \quad\text{or}\quad \{\alpha,h\alpha h,\alpha h,h\alpha\}\cap T_{i,j}=\emptyset, \qquad (4)$$

for $1\le i\le m-1$ and $1\le j\le n$.

Hence, let $1\le i\le m-1$ and $1\le j\le n$ and let $\alpha_1,\alpha_2\in\mathcal{OD}_{m\times n}$ be such that $\alpha_1\alpha_2\in T_{i,j}$. Next, we show that $\alpha_1\in T_{i,j}$ or $\alpha_2\in T_{i,j}$, by considering four cases, which finishes the proof.

CASE 1. If $\alpha_1,\alpha_2\in\mathcal{O}_{m\times n}$, then $\alpha_1\alpha_2\in Q_{i,j}\cup Q_{m-i,n-j+1}$ and so, by the observation at the start of the proof, we have $\alpha_1\in Q_{i,j}\cup Q_{m-i,n-j+1}$ or $\alpha_2\in Q_{i,j}\cup Q_{m-i,n-j+1}$, whence $\alpha_1\in T_{i,j}$ or $\alpha_2\in T_{i,j}$.

CASE 2. If $\alpha_1\notin\mathcal{O}_{m\times n}$ and $\alpha_2\notin\mathcal{O}_{m\times n}$, then $(\alpha_1 h)(h\alpha_2)=\alpha_1\alpha_2\in T_{i,j}$ and $\alpha_1 h, h\alpha_2\in\mathcal{O}_{m\times n}$ and so, by CASE 1, $\alpha_1 h\in T_{i,j}$ or $h\alpha_2\in T_{i,j}$. Hence, by (4), $\alpha_1\in T_{i,j}$ or $\alpha_2\in T_{i,j}$.

CASE 3. If $\alpha_1\notin\mathcal{O}_{m\times n}$ and $\alpha_2\in\mathcal{O}_{m\times n}$, then $h\alpha_1\in\mathcal{O}_{m\times n}$ and $\alpha_2\in\mathcal{O}_{m\times n}$ and, by (4), $(h\alpha_1)\alpha_2=h(\alpha_1\alpha_2)\in T_{i,j}$. Hence, by CASE 1, $h\alpha_1\in T_{i,j}$ or $\alpha_2\in T_{i,j}$ and so, again by (4), $\alpha_1\in T_{i,j}$ or $\alpha_2\in T_{i,j}$.

CASE 4. Finally, if $\alpha_1\in\mathcal{O}_{m\times n}$ and $\alpha_2\notin\mathcal{O}_{m\times n}$, then $\alpha_1\in\mathcal{O}_{m\times n}$ and $\alpha_2 h\in\mathcal{O}_{m\times n}$ and, by (4), $\alpha_1(\alpha_2 h)=(\alpha_1\alpha_2)h\in T_{i,j}$. Thus, by CASE 1, $\alpha_1\in T_{i,j}$ or $\alpha_2 h\in T_{i,j}$ and so, once again by (4), $\alpha_1\in T_{i,j}$ or $\alpha_2\in T_{i,j}$, as required. $\square$

Now, from Proposition 3.2 and Lemmas 3.3 and 3.4, the main result of this section follows immediately.

Theorem 3.5 *The rank of $\mathcal{OD}_{m\times n}$ is $\lceil\frac{mn}{2}\rceil+\lceil\frac{(m-1)n}{2}\rceil+1$.* $\square$

4 The rank of the monoid $\mathcal{OR}_{m\times n}$

As for $\mathcal{OD}_{m\times n}$, if $\alpha\in\mathcal{T}_{mn}$, then α is an orientation-reversing transformation if and only if $h\alpha$ (respectively, αh) is an orientation-preserving transformation. Hence, as $\alpha=h^2\alpha=h(h\alpha)$, it is clear that the monoid $\mathcal{OR}_{m\times n}$ is generated by $\mathcal{OP}_{m\times n}\cup\{h\}$. From Section 2, recall that $\{f,c_1,b_{1,1}\ldots,b_{1,n-1},s_1,t_{1,2}\ldots,t_{1,n},p_1,\ldots,p_{\lceil\frac{n-1}{2}\rceil}\}$ is a generating set, with $2n+\lceil\frac{n-1}{2}\rceil+1$ elements, of the monoid $\mathcal{OP}_{m\times n}$. Furthermore, for $m=2$, the set $\{f,c_1,b_{1,1}\ldots,b_{1,n-1},p_1,\ldots,p_{\lceil\frac{n-1}{2}\rceil}\}$ generates $\mathcal{OP}_{2\times n}$ and has just $n+\lceil\frac{n-1}{2}\rceil+1$ elements.

Now, for $1\le j\le\lceil\frac{n}{2}\rceil$, let v_j be the transformation of $\mathcal{O}_{m\times n}$ of rank $mn-1$, whose image is $\{1,\ldots,mn\}\setminus\{j\}$ and whose kernel is defined by the partition $\{\{1\},\ldots,\{\lceil\frac{n}{2}\rceil-j\},\{\lceil\frac{n}{2}\rceil-j+1,\lceil\frac{n}{2}\rceil-j+2\},\{\lceil\frac{n}{2}\rceil-j+3\},\ldots,\{mn\}\}$.

Example 4.1 For $m = 3$ and $n = 5$, we have:

$$v_1 = \left(\begin{array}{ccccc|ccccc|ccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 2 & 3 & 4 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \end{array} \right),$$

$$v_2 = \left(\begin{array}{ccccc|ccccc|ccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 1 & 3 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \end{array} \right),$$

$$v_3 = \left(\begin{array}{ccccc|ccccc|ccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 1 & 1 & 2 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \end{array} \right).$$

It is a routine matter to prove the following lemma.

Lemma 4.1 *The following equalities hold:*

1. $b_{1,j} = v_{\lceil \frac{n}{2} \rceil - j+1} v_j$, *for* $1 \le j \le \lceil \frac{n}{2} \rceil$;
2. $b_{1,n-j} = f^{m-1} h v_{\lceil \frac{n}{2} \rceil - j+1} v_{j+1} f^{m-1} h$, *for* $1 \le j \le n - \lceil \frac{n}{2} \rceil - 1$;
3. $c_1 = hfb_{1,n-1} b_{1,n-2} \cdots b_{1,2} b_{1,1} f^{m-1} h$;
4. $t_{1,n} = f^{m-2} h s_1 f^{m-2} h$;
5. $t_{1,j} = c_1^{n-j} hf^2 (b_{1,n-j+1} b_{1,n-j} \cdots b_{1,2})(b_{1,n-j+2} b_{1,n-j+1} \cdots b_{1,3}) \cdots (b_{1,n-1} b_{1,n-2} \cdots b_{1,j}) t_{1,n-j+1} s_1 f^{m-2} h$, *for* $2 \le j \le n-1$. $\square$

Therefore, it is easy to prove that:

Proposition 4.2 *The set* $\{f, s_1, t_{1,2} \ldots, t_{1,\lceil \frac{n}{2} \rceil}, p_1, \ldots, p_{\lceil \frac{n-1}{2} \rceil}, v_1, \ldots, v_{\lceil \frac{n}{2} \rceil}, h\}$ *has* $2\lceil \frac{n}{2} \rceil + \lceil \frac{n-1}{2} \rceil + 2$ *elements and generates* $\mathcal{OR}_{m\times n}$. *Furthermore, for* $m = 2$, *the set* $\{f, p_1, \ldots, p_{\lceil \frac{n-1}{2} \rceil}, v_1, \ldots, v_{\lceil \frac{n}{2} \rceil}, h\}$ *has* $\lceil \frac{n}{2} \rceil + \lceil \frac{n-1}{2} \rceil + 2$ *elements and generates* $\mathcal{OR}_{2\times n}$. $\square$

In what follows, we show that the first and second sets of the last result are a least size generating set of $\mathcal{OR}_{m\times n}$, for $m > 2$, and of $\mathcal{OP}_{2\times n}$, respectively.

First, notice that any generating set $\mathcal{OR}_{m\times n}$ must contain two distinct permutations of X_{mn}, one preserving the orientation and another reversing the orientation.

Next, we consider transformations of rank $mn - 1$.

Lemma 4.3 *Any generating set of* $\mathcal{OR}_{m\times n}$ *contains at least* $\lceil \frac{n}{2} \rceil$ *distinct elements of rank* $mn - 1$.

Proof. For each $1 \le t \le mn$, let $K_t = \{1, 2, \ldots, mn\} \backslash \{t\}$. Let U be a generating set of $\mathcal{OR}_{m\times n}$ and let $\xi_1, \ldots, \xi_k$ ($k \ge 1$) be all the elements of U of rank $mn - 1$. Then, for $1 \le j \le k$, we have that $\mathrm{Im}(\xi_j) = K_{\ell_j}$, for some $1 \le \ell_j \le mn$. For $1 \le j \le k$ and $1 \le i \le m-1$, define ℓ_{ik+j} as being the element of X_{mn} congruent modulo mn with $\ell_j + in$.

Now, take a transformation $\gamma \in \mathcal{OR}_{m\times n}$ of rank $mn - 1$. Then, $\gamma = \alpha \xi_j f^i$ or $\gamma = \alpha \xi_j f^i h$, for some $j \in \{1, \ldots, k\}$, $i \in \{0, \ldots, m-1\}$ and $\alpha \in \mathcal{OR}_{m\times n}$. Hence, $\mathrm{Im}(\gamma) = K_{\ell_{ik+j}}$ or $\mathrm{Im}(\gamma) = K_{mn-\ell_{ik+j}+1}$. As we have precisely mn possible distinct images for a transformation of $\mathcal{OR}_{m\times n}$ of rank $mn - 1$, the set $\{K_{\ell_1}, \ldots, K_{\ell_{mk}}, K_{mn-\ell_1+1}, \ldots, K_{mn-\ell_{mk}+1}\}$ has at least mn distinct elements. Thus $2mk \ge mn$ and so $k \ge \lceil \frac{n}{2} \rceil$, as required. $\square$

For the transformations of rank $(m-1)n$, we have:

Lemma 4.4 *For* $m > 2$, *any generating set of* $\mathcal{OR}_{m\times n}$ *contains at least* $\lceil \frac{n}{2} \rceil$ *distinct elements of rank* $(m-1)n$.

Proof. This proof is similar to Lemma 3.4 and so we omit some details.

For $j \in \{1, \ldots, n\}$, consider

$$T_j = \{\alpha \in \mathcal{OP}_{m\times n} \mid \mathrm{rank}(\alpha) = (m-1)n \text{ and } (kn)\alpha = (kn+1)\alpha = (i-1)n + j, \text{ for some } 1 \le i, k \le m\}.$$

Recall that, in the proof of Lemma 2.7, we showed that $T_1, \ldots, T_n$ are n two by two disjoint subsets of $\mathcal{OP}_{m\times n}$ such that, given $\alpha_1, \alpha_2 \in \mathcal{OP}_{m\times n}$, if $\alpha_1\alpha_2 \in T_j$ then $\alpha_1 \in T_j$ or $\alpha_2 \in T_j$, for $1 \le j \le n$. Moreover, it is easy to show that, given $\alpha \in \mathcal{T}_{mn}$, we have $\alpha \in T_j$ if and only if $h\alpha h \in T_{n-j+1}$ and, consequently, $h\alpha \in T_j$ if and only if $\alpha h \in T_{n-j+1}$, for $1 \le j \le n$. Define

$$U_j = \{\alpha \in \mathcal{OR}_{m\times n} \mid \alpha \in T_j \cup T_{n-j+1} \text{ or } h\alpha \in T_j \cup T_{n-j+1}\},$$

for $1 \leq j \leq n$.

First, observe that, clearly, $U_j = U_{n-j+1}$, for $1 \leq j \leq n$. Also, it is easy to show that, if $j, j' \in \{1, \ldots, n\}$ are such that $U_j \cap U_{j'} \neq \emptyset$ then $j' \in \{j, n-j+1\}$. It follows that $U_1, \ldots, U_{\lceil \frac{n}{2} \rceil}$ are $\lceil \frac{n}{2} \rceil$ two by two disjoint non-empty subsets of $\mathcal{OR}_{m\times n}$.

Secondly, notice that, given $\alpha \in \mathcal{T}_{mn}$, it is also easy to show that $\{\alpha, h\alpha h, \alpha h, h\alpha\} \subseteq U_j$ or $\{\alpha, h\alpha h, \alpha h, h\alpha\} \cap U_j = \emptyset$, for $1 \leq j \leq n$. Hence, it is a routine matter to prove that, for $1 \leq j \leq n$ and $\alpha_1, \alpha_2 \in \mathcal{OR}_{m\times n}$ such that $\alpha_1\alpha_2 \in U_j$, we have $\alpha_1 \in U_j$ or $\alpha_2 \in U_j$. It follows, by induction on k, that to write an element of U_j as a product of k elements of $\mathcal{OR}_{m\times n}$, we must have a factor that belongs to U_j, for $1 \leq j \leq \lceil \frac{n}{2} \rceil$, which proves the lemma. $\square$

Next, we deal with transformations of $\mathcal{OR}_{m\times n}$ of rank n. As for $\mathcal{OP}_{m\times n}$, we aim to show that, in order to generate $\mathcal{OR}_{m\times n}$, at least $\lceil \frac{n-1}{2} \rceil$ distinct transformations of rank n are required.

We begin with an observation, for which we need to introduce notation first. For each $n \in \mathbb{N}$, denote by h_n the reflexion permutation $\begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix}$ of X_n. Observe that, with this notation, we have $h = h_{mn}$ and, moreover, $h\psi = (h_n, h_n, \ldots, h_n; h_m)$. Furthermore, being $\alpha \in \mathcal{T}_{m\times n}$ and $\alpha\psi = (\alpha_1, \alpha_2, \ldots, \alpha_m; \beta)$, we obtain

$$(h\alpha h)\psi = (h_n\alpha_m h_n, h_n\alpha_{m-1}h_n, \ldots, h_n\alpha_1 h_n; h_m\beta h_m). \tag{5}$$

Notice that, clearly,

$$|\operatorname{Im}(h_m\beta h_m)| = |\operatorname{Im}(\beta)| \quad \text{and} \quad |\operatorname{Im}(h_n\alpha_i h_n)| = |\operatorname{Im}(\alpha_i)|, \tag{6}$$

for $1 \leq i \leq m$.

Now, recall the $\lceil \frac{n-1}{2} \rceil$ two by two disjoint subsets of $\mathcal{OP}_{m\times n}\psi$

$$P_i = \{(\gamma_1, \ldots, \gamma_m; \lambda) \in \overline{N} \mid |\operatorname{Im}(\gamma_k)| = n-i+1 \text{ and } |\operatorname{Im}(\gamma_\ell)| = i+1, \text{ for some } 1 \leq k, \ell \leq m \text{ such that } k \neq \ell\},$$

with $1 \leq i \leq \lceil \frac{n-1}{2} \rceil$, considered in the proof of Lemma 2.9. Given $\alpha \in \mathcal{T}_{mn}$, from (5) and (6), it follows immediately that

$$\alpha\psi \in P_i \text{ if and only if } (h\alpha h)\psi \in P_i \quad \text{and, consequently,} \quad (h\alpha)\psi \in P_i \text{ if and only if } (\alpha h)\psi \in P_i, \tag{7}$$

for $1 \leq i \leq \lceil \frac{n-1}{2} \rceil$.

Next, following the same strategy of Lemmas 3.4 and 4.4, we define

$$Q_i = \{\alpha \in \mathcal{OR}_{m\times n} \mid \alpha\psi \in P_i \text{ or } (h\alpha)\psi \in P_i\},$$

for $1 \leq i \leq \lceil \frac{n-1}{2} \rceil$.

First, observe that, as $P_1\psi^{-1}, \ldots, P_{\lceil \frac{n-1}{2} \rceil}\psi^{-1}$ are $\lceil \frac{n-1}{2} \rceil$ two by two disjoint subsets of transformations of rank n of $\mathcal{OP}_{m\times n}$, it is clear that also $Q_1, \ldots, Q_{\lceil \frac{n-1}{2} \rceil}$ are $\lceil \frac{n-1}{2} \rceil$ two by two disjoint subsets of transformations of rank n of $\mathcal{OR}_{m\times n}$.

On the other hand, from (7), we also deduce that

$$\{\alpha, h\alpha h, \alpha h, h\alpha\} \subseteq Q_i \quad \text{or} \quad \{\alpha, h\alpha h, \alpha h, h\alpha\} \cap Q_i = \emptyset, \tag{8}$$

for $\alpha \in \mathcal{T}_{mn}$ and $1 \leq i \leq \lceil \frac{n-1}{2} \rceil$. Now, recall we proved in Lemma 2.9 that $\alpha_1\alpha_2 \in P_i\psi^{-1}$ implies $\alpha_1 \in P_i\psi^{-1}$ or $\alpha_2 \in P_i\psi^{-1}$, for $\alpha_1, \alpha_2 \in \mathcal{OP}_{m\times n}$ and $1 \leq i \leq \lceil \frac{n-1}{2} \rceil$. Hence, by using properly the property (8), it is easy to show also that, given $\alpha_1, \alpha_2 \in \mathcal{OR}_{m\times n}$, if $\alpha_1\alpha_2 \in Q_i$ then $\alpha_1 \in Q_i$ or $\alpha_2 \in Q_i$, for $1 \leq i \leq \lceil \frac{n-1}{2} \rceil$. Thus, by induction on k, it follows that to write an element of Q_i as a product of k elements of $\mathcal{OR}_{m\times n}$, we must have a factor that belongs to Q_i, for $1 \leq i \leq \lceil \frac{n-1}{2} \rceil$.

Therefore, we have proved that:

Lemma 4.5 *Any generating set of $\mathcal{OR}_{m\times n}$ contains at least $\lceil \frac{n-1}{2} \rceil$ distinct elements of rank n.* $\square$

Finally, it follows our main objective of this section.

Theorem 4.6 *The rank of $\mathcal{OR}_{m\times n}$ is equal to $2\lceil \frac{n}{2} \rceil + \lceil \frac{n-1}{2} \rceil + 2$, for $m > 2$, and equal to $\lceil \frac{n}{2} \rceil + \lceil \frac{n-1}{2} \rceil + 2$, for $m = 2$.*

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